TENSOR PRODUCT VARIETIES AND CRYSTALS
ADE CASE

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ABSTRACT. Let $\mathfrak{g}$ be a simple simply laced Lie algebra. In this paper two families of varieties associated to the Dynkin graph of $\mathfrak{g}$ are described: "tensor product" and "multiplicity" varieties. These varieties are closely related to Nakajima's quiver varieties and should play an important role in the geometric constructions of tensor products and intertwining operators. In particular it is shown that the set of irreducible components of a tensor product variety can be equipped with a structure of $\mathfrak{g}$-crystal isomorphic to the crystal of the canonical basis of the tensor product of several simple finite dimensional representations of $\mathfrak{g}$, and that the number of irreducible components of a multiplicity variety is equal to the multiplicity of a certain representation in the tensor product of several others. Moreover the decomposition of a tensor product into a direct sum is described geometrically (on the level of crystals).

0. INTRODUCTION

0.0.1.   The purpose of this paper is to define and study two (closely related) families of quasi-projective varieties associated to a simply laced Dynkin graph D: the *tensor product varieties* and the *multiplicity varieties*.

0.0.2.   In this paper for the sake of clarity the Dynkin graph D is assumed to be of ADE type. This condition can be relaxed by adjusting slightly the definitions of the varieties involved.

Let $\mathfrak{g}$ be the simple Lie algebra associated to D and $\mathfrak{g}' = \mathfrak{g} \oplus \mathfrak{t}$, where $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. Thus $\mathfrak{g}'$ is a reductive Lie algebra.

The precise definition of the tensor product and the multiplicity varieties is rather involved (see 2.6). For the purpose of this Introduction it is enough to know that a multiplicity variety ${}^n\mathfrak{S}(\mu^0; \mu^1, \dots, \mu^n)$ is associated to a set $\mu^0, \mu^1, \dots, \mu^n$, where μ^k belongs to a certain subset of the weight lattice of the reductive Lie algebra $\mathfrak{g}'$ (the set of "integrable positive" weights). A tensor product variety ${}^n\mathfrak{T}(\mu^1, \dots, \mu^n; \nu)$ is associated to a set $\mu^1, \dots, \mu^n, \nu$, where $\mu^1, \dots, \mu^n$ are as above, and ν is a weight of $\mathfrak{g}'$. This way of writing parameters of ${}^n\mathfrak{S}$ and ${}^n\mathfrak{T}$ is for the purposes of the Introduction only. Notation in the main body of the paper is different.

In the case $n = 1$ the variety ${}^1\mathfrak{S}(\mu^0; \mu^1)$ is empty unless $\mu^0 = \mu^1$ in which case it coincides with Nakajima's quiver variety $\mathfrak{M}_0^{reg}(v, w)$ (cf. [Nak94, Nak98]). Here v and w can be expressed in terms of μ^0. Similarly ${}^1\mathfrak{T}(\mu^1; \nu)$ coincides with a fiber $\mathfrak{M}(\mu^1, \nu) = \mathfrak{M}(v, v_0, w)$ of Nakajima's resolution of singularities of the singular quiver variety (where again v, v_0, and w can be expressed in terms of μ^1 and ν).

0.0.3. Multiplicity varieties and tensor product varieties have pure dimensions. Let $^n\mathcal{T}(\mu^1,\dots,\mu^n;\nu)$ (resp. $^n\mathcal{S}(\mu^0;\mu^1,\dots,\mu^n)$, $\mathcal{M}(\mu,\nu)$) be the set of irreducible components of the tensor product variety $^n\mathfrak{T}(\mu^1,\dots,\mu^n;\nu)$ (resp. the multiplicity variety $^n\mathfrak{S}(\mu^0;\mu^1,\dots,\mu^n)$, Nakajima's variety $\mathfrak{M}(\mu,\nu)$), and let $^n\mathcal{T}(\mu^1,\dots,\mu^n) = \bigsqcup_\nu {}^n\mathcal{T}(\mu^1,\dots,\mu^n;\nu)$, $\mathcal{M}(\mu) = \bigsqcup_\nu \mathcal{M}(\mu,\nu)$.

Nakajima [Nak98] (based on an idea of Lusztig [Lus91, 12]) introduced a structure of a $\mathfrak{g}'$-crystal on the set $\mathcal{M}(\mu)$ and it was shown by Saito (based on his joint work with Kashiwara [KS97]) that this crystal is isomorphic to the crystal of the canonical basis of the irreducible representation of $\mathfrak{g}'$ with highest weight μ. Strictly speaking the above mentioned authors consider $\mathfrak{g}$-crystals, but the extension to $\mathfrak{g}'$ is straightforward and the weight lattice of $\mathfrak{g}'$ appears more naturally in geometry of quiver varieties.

The main result of this paper is the construction of two bijections between sets of irreducible components (cf. 2.10, 2.13):

$$\alpha_n: \quad ^n\mathcal{T}(\mu^1,\dots,\mu^n) \xrightarrow{\sim} \mathcal{M}(\mu_1) \times \dots \times \mathcal{M}(\mu_n)\,,$$

$$\beta_n: \quad ^n\mathcal{T}(\mu^1,\dots,\mu^n) \xrightarrow{\sim} \bigsqcup_{\mu_0} {}^n\mathcal{S}(\mu^0;\mu^1,\dots,\mu^n) \times \mathcal{M}(\mu^0)\,.$$

Moreover one has the following theorem (cf. Theorems 3.9, 3.11).

Theorem. *The composite bijection*

$$\tau_n = \beta_n \circ \alpha_n^{-1}: \quad \mathcal{M}(\mu_1) \otimes \dots \otimes \mathcal{M}(\mu_n) \xrightarrow{\sim} \bigoplus_{\mu_0} {}^n\mathcal{S}(\mu^0;\mu^1,\dots,\mu^n) \otimes \mathcal{M}(\mu^0)$$

is an isomorphism of $\mathfrak{g}'$-crystals if one endows the sets $\mathcal{M}(\mu_1),\dots,\mathcal{M}(\mu_n)$ with Nakajima's crystal structure, and considers the set $^n\mathcal{S}(\mu^0;\mu^1,\dots,\mu^n)$ as a trivial crystal.

In the above theorem $\otimes$ (resp. $\oplus$) denotes the tensor product (resp. direct sum) of crystals, which coincides with the direct product (resp. disjoint union) on the level of sets.

It follows in particular, that the set $^n\mathcal{T}(\mu^1,\dots,\mu^n)$ can be equipped with the structure of a $\mathfrak{g}'$-crystal using ether of the bijections α_n or β_n, and this crystal is isomorphic to the crystal of the canonical basis in the tensor product of n irreducible representations of $\mathfrak{g}'$ with highest weights $\mu^1,\dots,\mu^n$.

0.0.4. The proof of the Theorem 0.0.3 uses double reduction. First it is shown that the statement for any n follows from the corresponding statement for $n = 2$. Then one uses restriction to Levi factors of parabolic subalgebras of $\mathfrak{g}$ to reduce the problem to $\mathfrak{sl}_2$ case. When $n = 2$ and $\mathfrak{g} = \mathfrak{sl}_2$ the Theorem becomes an elementary linear algebraic statement.

0.0.5. The existence of the crystal isomorphism τ_n allows one to apply a theorem of Joseph [Jos95, Proposition 6.4.21] about the uniqueness of the family of crystals closed with respect to tensor products to give another proof of isomorphism between $\mathcal{M}(\mu)$ and the crystal of the canonical basis of the irreducible representation of $\mathfrak{g}'$ with highest weight μ, and, more importantly, to prove that the set $^n\mathcal{S}(\mu^0;\mu^1,\dots,\mu^n)$ of irreducible components of a multiplicity variety has the cardinal equal to the multiplicity of the irreducible representation of $\mathfrak{g}'$ with highest weight μ^0 in a direct sum decomposition of the tensor product of n irreducible representations with highest weights $\mu^1,\dots,\mu^n$. It is also equal to the corresponding

multiplicity for the tensor product of representations of the simple Lie algebra $\mathfrak{g}$ if one restricts the weights to the Cartan subalgebra of $\mathfrak{g}$.

In other words one has the following corollary of the Theorem 0.0.3.

Corollary. *Let $L(\mu)$ denote the irreducible representation of $\mathfrak{g}'$ with highest weight μ. Then*

$$|{}^n\mathcal{S}(\mu^0; \mu^1, \ldots, \mu^n)| = \dim_{\mathbb{C}} \operatorname{Hom}_{\mathfrak{g}'}\left(L(\mu^0), L(\mu^1) \otimes \ldots \otimes L(\mu^n)\right).$$

0.0.6. The paper is organized as follows. Section 1 contains a review of Kashiwara's theory of crystals. Section 2 begins with a description of Nakajima's quiver varieties. Then the tensor product and the multiplicity varieties are defined in 2.6. The reminder of Section 2 is devoted to various properties of these varieties. In particular the tensor decomposition bijection is described in 2.14. In Section 3 it is shown (following Lusztig and Nakajima) how one can use restrictions to Levi subalgebras of parabolic subalgebras of $\mathfrak{g}$ to define crystal structures on the sets of irreducible components of the varieties involved. Then a variant of Theorem 0.0.3 (Theorem 3.9) is proven using the Levi restriction.

0.0.7. Definitions of various varieties given in the main body of the paper are rather messy due to abundance of indices. Unfortunately, one needs this notation in order to follow the arguments used in the proofs. However in the Appendix a more conceptual definition of the multiplicity varieties is given. Namely they appear to be closely related to the Hall-Ringel algebra (cf. [Rin90]) associated to a certain algebra $\tilde{\mathcal{F}}$ introduced by Lusztig [Lus98, Lus00a]. The Appendix also contains a diagram of varieties called the tensor product diagram which can be (conjecturally) used to equip a certain category of perverse sheaves on a variety Z_D described by Lusztig [Lus98] with a structure of a Tannakian category.

0.0.8. Tensor product varieties ${}^n\mathfrak{T}(\mu^1, \ldots, \mu^n; v)$ are also described in a recent paper of H. Nakajima [Nak01b]. Actually, Nakajima only considers some special values of the highest weights $\{\mu^i\}$ that correspond to Lagrangian fibers in quiver varieties. However it is not restrictive for representation theory applications, and generalization to arbitrary $\{\mu^i\}$ is straightforward.

Nakajima studies deeper geometric structures on a tensor product variety than just the set of its irreducible components, namely the Borel-Moore homology and equivariant K-theory, which allows him to obtain some very interesting results in the representation theory of quantum affine algebras at zero level.

On the contrary the author of the present paper is mainly interested in the geometric description of the multiplicities in tensor products, and more generally in monoidal structures on various geometric objects on the singular quiver variety Z (cf. the Appendix). The importance of tensor product varieties for him is in their relation to resolutions of singularities of multiplicity varieties.

0.0.9. Various special cases of tensor product and multiplicity varieties have appeared in the literature before (and certainly served as important sources of motivation for the author). The most important case is related to a Dynkin graph of type A. It is known (cf. [Nak94]) that some special cases of quiver varieties associated to this Dynkin graph are related to partial resolution of singularities of the nilpotent cone in $\mathfrak{gl}_k$. Similarly, the multiplicity varieties in this case are related to Spaltenstein varieties for $\mathfrak{gl}_k$ (a Spaltenstein variety is a variety consisting

of all parabolic subgroups P of a given type that contain a given nilpotent operator $t \in \mathfrak{gl}_k$, and such that the projection of t to the Levi factor L of P belongs to a given nilpotent orbit in L). A theorem due to Hall (cf. [Hal59], [Mac95, Chapter II]) implies that the number of irreducible components of a Spaltenstein variety is equal to a certain multiplicity in the tensor product decomposition for $\mathfrak{gl}_N$ (k has no relation to N), which is a special case of Corollary 0.0.5. To the best of the author's knowledge the tensor product varieties are new even in the $\mathfrak{gl}_N$-case. All known proofs of the Hall theorem are either combinatorial (through the Littlewood-Richardson Rule), or use the relation between the tensor product for GL and the restriction for symmetric groups combined with results of Borho and MacPherson [BM83]. The existence of the tensor product varieties together with bijections α_n and β_n provides a direct proof of the Hall theorem by showing the role of Spaltenstein varieties in the geometric theory of the tensor product.

In the special case of the tensor product of n fundamental representations of $\mathfrak{gl}_N$ the tensor product variety is a certain Lagrangian subvariety in the cotangent bundle of the variety studied by Grojnowski and Lusztig in [GL92]. However in general tensor product varieties cannot be represented in this form (and thus cannot be used to prove positivity properties of canonical bases).

The $\mathfrak{gl}_N$ case is studied in [Mal00] without mentioning quiver terminology (purely in the language of flags and nilpotent orbits).

0.0.10.   Recently Lusztig [Lus00a] described a locally closed subset of the variety Z (no relation with Z_D of 0.0.7) constructed by Nakajima (cf. [Nak98]), such that irreducible components of this subset (conjecturally) form a crystal isomorphic to the tensor product of two irreducible representations of $\mathfrak{g}$. The relation of this construction to the tensor product variety (for $n = 2$) will be discussed elsewhere. In particular, using constructions of this paper one can show that the set of irreducible components of the Lusztig's variety is in a natural bijection with the set of irreducible components of the corresponding tensor product variety. Note however that Lusztig's construction does not produce a direct sum decomposition for a tensor product, nor it can be generalized to a case of more than two multiples. On the other hand it provides a geometric analogue of a factor of the universal enveloping algebra of $\mathfrak{g}$ isomorphic to the tensor product of a highest weight and a lowest weight representations.

0.0.11.   An important source of inspiration for the author was a work of A. Braverman and D. Gaitsgory [BG99] where they constructed geometric crystals (together with a crystal tensor product) via the Affine Grassmannian. Note that Nakajima's construction of crystals (cf. 0.0.3) uses a fiber of a resolution of singularities of a singular variety (quiver variety $\mathfrak{M}_0(v, w)$), while Braverman and Gaitsgory use cells of a perverse stratification of a singular variety (the Affine Grassmannian). Similar difference exists between the constructions of the crystal tensor product.

0.0.12.   Though the ground field is $\mathbb{C}$ throughout the paper the multiplicity varieties are defined over arbitrary fields, and it is shown in [Mal01a] that the number of $\mathbb{F}_q$-rational points of ${}^n\mathcal{S}(\mu^0; \mu^1, \dots, \mu^n)$ is given by a polynomial in q with the leading coefficient equal to $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}'}(L(\mu^0), L(\mu^1) \otimes \dots \otimes L(\mu^n))$ (cf. Corollary 0.0.5). This statement is a direct generalization of the Hall theorem (cf. [Hal59], [Mac95, Chapter II]) giving the number of $\mathbb{F}_q$-rational points in a Spaltenstein variety for $\mathfrak{gl}_N$.

0.0.13. Throughout the paper the following conventions are used: the ground field is $\mathbb{C}$; "closed", "locally closed", etc., refer to the Zariski topology; "fibration" means "locally trivial fibration", where "locally" refers to the Zariski topology, however trivialization is analytic (not regular).

0.0.14. *Acknowledgements.* This paper owns its very existence to Igor Frenkel and Lev Rozansky. All ideas described below crystallized during our numerous discussions in the summer 2000.

CONTENTS

1. CRYSTALS

Crystals were unearthed by M. Kashiwara [Kas90, Kas91, Kas94]. An excellent survey of crystals as well as some new results are given by A. Joseph in [Jos95, Chapters 5, 6].

1.1. **Weights and roots.** Let $\mathfrak{g}$ be a reductive or a Kac-Moody Lie algebra, I be the set of vertices of the Dynkin graph of $\mathfrak{g}$. It is assumed throughout the paper that $\mathfrak{g}$ is simply laced. The weight and coweight lattices are identified with $\mathbb{Z}[I]$, so that the natural pairing between them (denoted $<,>$) becomes

$$< v, u >= \sum_{i \in I} v_i u_i \ ,$$

where v_i denotes the i-th component of $v \in \mathbb{Z}[I]$.

Let A be the Cartan matrix of $\mathfrak{g}$. Then the simple root $\hat{i}$, corresponding to a simple weight $i \in I$ is given by $Ai = \sum_{j \in I} A_{ji} j \in \mathbb{Z}_{\geq 0}[I]$.

1.2. **Definition of $\mathfrak{g}$-crystals.**

A $\mathfrak{g}$-*crystal* is a tuple $(\mathcal{A}, wt, \{\varepsilon_i\}_{i \in I}, \{\varphi_i\}_{i \in I}, \{\tilde{e}_i\}_{i \in I}, \{\tilde{f}_i\}_{i \in I})$, where

- $\mathcal{A}$ is a set,
- wt is a map from $\mathcal{A}$ to $\mathcal{Q}_{\mathfrak{g}}$,
- ε_i and φ_i are maps from $\mathcal{A}$ to $\mathbb{Z}$,
- $\tilde{e}_i$ and $\tilde{f}_i$ are maps from $\mathcal{A}$ to $\mathcal{A} \cup \{0\}$.

These data should satisfy the following axioms:

- $wt(\tilde{e}_i a) = wt(a) + \hat{i}, \quad \varphi_i(\tilde{e}_i a) = \varphi_i(a) + 1, \quad \varepsilon_i(\tilde{e}_i a) = \varepsilon_i(a) - 1, \quad$ for any $i \in I$ and $a \in \mathcal{A}$ such that $\tilde{e}_i a \neq 0$,
- $wt(\tilde{f}_i a) = wt(a) - \hat{i}, \quad \varphi_i(\tilde{f}_i a) = \varphi_i(a) - 1, \quad \varepsilon_i(\tilde{f}_i a) = \varepsilon_i(a) + 1, \quad$ for any $i \in I$ and $a \in \mathcal{A}$ such that $\tilde{f}_i a \neq 0$,
- $(wt(a))_i = \varphi_i(a) - \varepsilon_i(a)$ for any $i \in I$ and $a \in \mathcal{A}$,
- $\tilde{f}_i a = b$ if and only if $\tilde{e}_i b = a$, where $i \in I$, and $a, b \in \mathcal{A}$.

The maps $\tilde{e}_i$ and $\tilde{f}_i$ are called Kashiwara's operators, and the map wt is called the weight function.

Remark. In a more general definition of crystals the maps ε_i and φ_i are allowed to have infinite values.

A $\mathfrak{g}$-crystal $(\mathcal{A}, wt, \{\varepsilon_i\}_{i \in I}, \{\varphi_i\}_{i \in I}, \{\tilde{e}_i\}_{i \in I}, \{\tilde{f}_i\}_{i \in I})$ is called *trivial* if

$$wt(a) = 0 \text{ for any } a \in \mathcal{A},$$

$$\varepsilon_i(a) = \varphi_i(a) = 0 \text{ for any } i \in I \text{ and } a \in \mathcal{A},$$

$$\tilde{e}_i a = \tilde{f}_i a = 0 \text{ for any } i \in I \text{ and } a \in \mathcal{A}.$$

Any set $\mathcal{A}$ can be equipped with the trivial crystal structure as above.

A $\mathfrak{g}$-crystal $(\mathcal{A}, wt, \{\varepsilon_i\}_{i \in I}, \{\varphi_i\}_{i \in I}, \{\tilde{e}_i\}_{i \in I}, \{\tilde{f}_i\}_{i \in I})$ is called *normal* if

$$\varepsilon_i(a) = \max\{n \mid \tilde{e}_i^n a \neq 0\} \ ,$$

$$\varphi_i(a) = \max\{n \mid \tilde{f}_i^n a \neq 0\} \ ,$$

for any $i \in I$ and $a \in \mathcal{A}$. Thus in a normal $\mathfrak{g}$-crystal the maps ε_i and φ_i are uniquely determined by the action of $\tilde{e}_i$ and $\tilde{f}_i$. A trivial crystal is normal.

In the rest of the paper all $\mathfrak{g}$-crystals are assumed normal, and thus the maps ε_i and φ_i are usually omitted.

By abuse of notation a $\mathfrak{g}$-crystal $(\mathcal{A}, wt, \{\varepsilon_i\}_{i\in I}, \{\varphi_i\}_{i\in I}, \{\tilde{e}_i\}_{i\in I}, \{\tilde{f}_i\}_{i\in I})$ is sometimes denoted simply by $\mathcal{A}$.

An *isomorphism* of two $\mathfrak{g}$-crystals $\mathcal{A}$ and $\mathcal{B}$ is a bijection between the sets $\mathcal{A}$ and $\mathcal{B}$ commuting with the action of the operators $\tilde{e}_i$ and $\tilde{f}_i$, and the functions wt, ε_i, and φ_i.

The *direct sum* $\mathcal{A} \oplus \mathcal{B}$ of two $\mathfrak{g}$-crystals $\mathcal{A}$ and $\mathcal{B}$ is their disjoint union as sets with the maps $\tilde{e}_i$, $\tilde{f}_i$, wt, ε_i, and φ_i acting on each component of the union separately.

1.3. Tensor product of $\mathfrak{g}$-crystals.

The *tensor product* $\mathcal{A} \otimes \mathcal{B}$ of two $\mathfrak{g}$-crystals $\mathcal{A}$ and $\mathcal{B}$ is their direct product as sets equipped with the following crystal structure:

$$wt((a,b)) = wt(a) + wt(b) \, ,$$
$$\varepsilon_i((a,b)) = \max\{\varepsilon_i(a), \varepsilon_i(a) + \varepsilon_i(b) - \varphi_i(a)\} \, ,$$
$$\varphi_i((a,b)) = \max\{\varphi_i(b), \varphi_i(a) + \varphi_i(b) - \varepsilon_i(b)\} \, ,$$

(1.3.a)
$$\tilde{e}_i((a,b)) = \begin{cases} (\tilde{e}_i a \, , \, b) & \text{if } \varphi_i(a) \geq \varepsilon_i(b) \, , \\ (a \, , \, \tilde{e}_i b) & \text{if } \varphi_i(a) < \varepsilon_i(b) \, , \end{cases}$$

$$\tilde{f}_i((a,b)) = \begin{cases} (\tilde{f}_i a \, , \, b) & \text{if } \varphi_i(a) > \varepsilon_i(b) \, , \\ (a \, , \, \tilde{f}_i b) & \text{if } \varphi_i(a) \leq \varepsilon_i(b) \, . \end{cases}$$

Here $(a,0)$, $(0,b)$, and $(0,0)$ are identified with 0. One can check that the set $\mathcal{A} \times \mathcal{B}$ with the above structure satisfies all the axioms of a (normal) crystal, and that the tensor product of crystals is associative.

Remark. Tensor product of crystals is not commutative.

Remark. The tensor product (in any order) of a crystal $\mathcal{A}$ with a trivial crystal $\mathcal{B}$ is isomorphic to the direct sum $\bigoplus_{b\in\mathcal{B}} \mathcal{A}_b$ where each $\mathcal{A}_b$ is isomorphic to $\mathcal{A}$.

1.4. Highest weight crystals and closed families.

A crystal $\mathcal{A}$ is a *highest weight* crystal with highest weight $\lambda \in \mathcal{Q}_{\mathfrak{g}}$ if there exists an element $a_\lambda \in \mathcal{A}$ such that:

- $wt(a_\lambda) = \lambda$,
- $\tilde{e}_i a_\lambda = 0$ for any $i \in I$,
- any element of $\mathcal{A}$ can be obtained from a_λ by successive applications of the operators $\tilde{f}_i$.

Consider a family of highest weight normal crystals $\{\mathcal{A}(\lambda)\}_{\lambda\in\mathcal{J}}$ labeled by a set $\mathcal{J} \subset \mathcal{Q}_{\mathfrak{g}}$ (the highest weight of $\mathcal{A}(\lambda)$ is λ). The family $\{\mathcal{A}(\lambda)\}_{\lambda\in\mathcal{J}}$ is called *strictly closed* (with respect to tensor products) if the tensor product of any two members of the family is isomorphic to a direct sum of members of the family:

$$\mathcal{A}(\mu^1) \otimes \mathcal{A}(\mu^2) = \bigoplus_{\lambda\in\mathcal{J}} \mathcal{U}((\mu^1, \mu^2), \lambda) \otimes \mathcal{A}(\lambda) \, ,$$

where $\mathcal{U}((\mu^1, \mu^2), \lambda)$ is a set equipped with the trivial crystal structure. The family $\{\mathcal{A}(\lambda)\}_{\lambda\in\mathcal{J}}$ is called *closed* if the tensor product $\mathcal{A}(\mu^1) \otimes \mathcal{A}(\mu^2)$ of any two members of the family contains $\mathcal{A}_{\mu^1+\mu^2}$ as a direct summand. Any strictly closed family is closed. The converse is also true, as a corollary of Theorem 1.5.

Let $\mathcal{Q}_{\mathfrak{g}}^+ \subset \mathcal{Q}_{\mathfrak{g}}$ be the set of highest weights of integrable highest weights modules of $\mathfrak{g}$ (in the reductive case a module is called integrable if it is derived from a polynomial representation of the corresponding connected simply connected reductive

group, in the Kac-Moody case the highest weight should be a positive linear combination of the fundamental weights). The original motivation for the introduction of crystals was the discovery by M. Kashiwara [Kas91] and G. Lusztig [Lus91] of canonical (or crystal) bases in integrable highest weight modules of a (quantum) Kac-Moody algebra. These bases have many favorable properties, one of which is that as sets they are equipped with a crystal structure. In other words to each irreducible integrable highest weight module $L(\lambda)$ corresponds a normal crystal $\mathcal{L}(\lambda)$ (crystal of the canonical basis). In this way one obtains a strictly closed family of crystals $\{\mathcal{L}(\lambda)\}_{\lambda \in \mathcal{Q}_\mathfrak{g}^+}$, satisfying the following two properties:

- the cardinality of $\mathcal{L}(\lambda)$ is equal to the dimension of $L(\lambda)$;
- one has the following tensor product decompositions for $\mathfrak{g}$-modules and $\mathfrak{g}$-crystals:

$$L(\mu^1) \otimes L(\mu^2) = \bigoplus_{\lambda \in \mathcal{Q}_\mathfrak{g}^+} C((\mu^1, \mu^2), \lambda) \otimes L(\lambda) \,,$$

$$\mathcal{L}(\mu^1) \otimes \mathcal{L}(\mu^2) = \bigoplus_{\lambda \in \mathcal{Q}_\mathfrak{g}^+} \mathcal{C}((\mu^1, \mu^2), \lambda) \otimes \mathcal{L}(\lambda) \,,$$

where the cardinality of the set (trivial crystal) $\mathcal{C}((\mu^1, \mu^2), \lambda)$ is equal to the dimension of the linear space (trivial $\mathfrak{g}$-module) $C((\mu^1, \mu^2), \lambda)$.

The aim of this paper is to construct another strictly closed family of $\mathfrak{g}$-crystals $\{\mathcal{M}(\lambda)\}_{\lambda \in \mathcal{Q}_\mathfrak{g}^+}$ (for $\mathfrak{g}$ being gl_N or a symmetric Kac-Moody algebra), using geometry associated to $\mathfrak{g}$. The following crucial theorem ensures that this family is isomorphic to the family $\{\mathcal{L}(\lambda)\}_{\lambda \in \mathcal{Q}_\mathfrak{g}^+}$ of crystals of canonical bases (two families of crystals labeled by the same index set are called isomorphic if the corresponding members of the families are isomorphic as crystals).

1.5. Theorem. *There exists a unique (up to an isomorphism) closed family of $\mathfrak{g}$-crystals labeled by $\mathcal{Q}_\mathfrak{g}^+$.*

Proof. For the proof in Kac-Moody case see [Jos95, Proposition 6.4.21]. The statement for a reductive $\mathfrak{g}$ easily follows from the statement for the factor of $\mathfrak{g}$ by its center.   □

2. Nakajima varieties and tensor product varieties

2.1. **Oriented graphs and path algebras.** Let I be the set of vertices of the Dynkin graph of a simple simply laced Lie algebra $\mathfrak{g}$. Let H be the set of pairs consisting of an edge of the Dynkin graph of $\mathfrak{g}$ and an orientation of this edge. The target (resp. source) vertex of $h \in H$ is denoted by $\mathrm{In}(h)$ (resp. $\mathrm{Out}(h)$). Thus (I, H) is an oriented graph (note that it has twice as many edges as the Dynkin graph of $\mathfrak{g}$ has). For $h \in H$ let $\bar{h}$ be the same edge but with opposite orientation (i.e. $\bar{h}$ is the unique element of H such that $\mathrm{In}(\bar{h}) = \mathrm{Out}(h)$ and $\mathrm{Out}(\bar{h}) = \mathrm{In}(h)$).

Let X be a symmetric $I \times I$ matrix uniquely defined by the following equation

$$(2.1.a) \qquad\qquad < Xv, u > = \sum_{h \in H} v_{\mathrm{In}(h)} u_{\mathrm{Out}(h)} \,,$$

for any v, $u \in \mathbb{Z}[I]$. Note that $X = 2\,\mathrm{Id} - A$, where A is the Cartan matrix of $\mathfrak{g}$.

Let $\mathcal{F}$ be the path algebra of the oriented graph (I, H) over $\mathbb{C}$. Fix a function $\varepsilon : H \to \mathbb{C}^*$ such that $\varepsilon(h) + \varepsilon(\bar{h}) = 0$ for any $h \in H$. Let

$$(2.1.b) \qquad \theta_i = \sum_{\substack{h \in H \\ \mathrm{In}(h)=i}} \varepsilon(h) h \bar{h} \in \mathcal{F} \,,$$

and let $\mathcal{F}_0$ be the factor algebra of $\mathcal{F}$ by the two-sided ideal generated by elements θ_i for all $i \in I$. The algebra $\mathcal{F}_0$ is called the *preprojective algebra*. It was introduced by Gelfand and Ponomarev.

Note that an $\mathcal{F}$-module is just a $\mathbb{Z}[I]$-graded $\mathbb{C}$-linear space V together with a collection of linear maps $x = \{x_h \in \mathrm{Hom}_{\mathbb{C}}(V_{\mathrm{Out}(h)}, V_{\mathrm{In}(h)})\}_{h \in H}$. This $\mathcal{F}$-module is denoted by (V, x). It is always assumed below that V is finite dimensional.

Given a finite dimensional $\mathbb{Z}[I]$-graded $\mathbb{C}$-linear space V, a set of linear maps $x = \{x_h \in \mathrm{Hom}_{\mathbb{C}}(V_{\mathrm{Out}(h)}, V_{\mathrm{In}(h)})\}_{h \in H}$ such that (V, x) is an $\mathcal{F}$-module is called a *representation* of $\mathcal{F}$ in V.

Let (V, x) be an $\mathcal{F}$-module and W be a $\mathbb{Z}[I]$-graded subspace of V such that $x_h W_{\mathrm{Out}(h)} \subset W_{\mathrm{In}(h)}$ for any $h \in H$. Then $(W, \, x|_w)$ (resp. $(V/W, \, x|_{V/W})$) is an $\mathcal{F}$-submodule (resp. factor module) of (V, x). Sometimes just W (resp. V/W) is called a submodule (resp. factor module) because the representation of $\mathcal{F}$ is uniquely defined by the restriction.

An $\mathcal{F}$-module (V, x) is also an $\mathcal{F}_0$-module if and only if $\sum_{\substack{h \in H \\ \mathrm{In}(h)=i}} \varepsilon(h) x_h x_{\bar{h}} = 0$ for any $i \in I$.

A k-tuple $h_1, h_2, \ldots, h_k$ of elements of H is called a *path* of length k if $\mathrm{In}(h_i) = \mathrm{Out}(h_{i-1})$ for $i = 2, \ldots, k$. An $\mathcal{F}$-module (V, x) is called *nilpotent* if there exists $N \in \mathbb{Z}_{>0}$ such that $x_{h_1} x_{h_2} \ldots x_{h_N} = 0$ for any path $h_1, h_2, \ldots, h_N$ of length N.

The following proposition is due to Lusztig [Lus91].

Proposition.

2.1.c. *An $\mathcal{F}$-module (V, x) is nilpotent if and only if $x_{h_1} x_{h_2} \ldots x_{h_{|\dim V|}} = 0$ (i.e. any path of length $|\dim V|$ is in the kernel of the representation x).*

2.1.d. *Any $\mathcal{F}_0$-module is nilpotent as an $\mathcal{F}$-module..*

Proof. The "if" part of 2.1.c follows from the definition of a nilpotent $\mathcal{F}$-module.

The "only if" part of 2.1.c is proved in [Lus91, Lemma 1.8].

2.1.d is proved in [Lus91, Section 12]. Recall that the Dynkin graph is assumed to be of finite (ADE) type.

$\square$

2.2. **Quiver varieties.** Quiver varieties were introduced by Nakajima [Nak94, Nak98] as a generalization of the moduli space of Yang-Mills instantons on an ALE space (cf. [KN90]). On the other hand they also generalize Spaltenstein variety for $GL(N)$ (the variety of all parabolic subgroups P such that a given unipotent element $u \in GL(N)$ belongs to the unipotent radical of P). Nakajima has shown in [Nak94, Nak98, Nak01a] that the quiver varieties play the same role in the representation theory of (quantum) Kac-Moody algebras as Spaltenstein varieties do in the representation theory of $GL(N)$ (cf. [Gin91, KL87]).

Let D, V be finite dimensional $\mathbb{Z}[I]$-graded $\mathbb{C}$-linear spaces. An *ADHM datum*[1] (on D, V) is a triple (x, p, q), where

$x = \{x_h\}_{h \in H}$ is a representation of $\mathcal{F}$ in V,

---

[1] ADHM stands for Atiyah, Drinfeld, Hitchin, Manin [AHDM78]

$p \in \mathrm{Hom}_{\mathbb{C}^I}(D, V)$ is a $\mathbb{Z}[I]$-graded $\mathbb{C}$-linear map from D to V,

$q \in \mathrm{Hom}_{\mathbb{C}^I}(V, D)$ is a $\mathbb{Z}[I]$-graded $\mathbb{C}$-linear map from V to D,

and x, p, q satisfy the following equation (in $\mathrm{End}_{\mathbb{C}} V_i$)

$$(2.2.a) \qquad \sum_{\substack{h \in H \\ \mathrm{In}(h)=i}} \varepsilon(h) x_h x_{\bar{h}} - p_i q_i = 0$$

for all $i \in I$. The set of all ADHM data on D, V form an affine variety denoted by $\Lambda_{D,V}$.

Let $D = \{0\}$. Then $p = q = 0$ and x gives a representation of the preprojective algebra $\mathcal{F}_0$ in V. The variety $\Lambda_{\{0\},V}$ is denoted simply by Λ_V, and an element $(x, 0, 0)$ of Λ_V is usually written simply as x. Thus Λ_V is the variety of all representations of the preprojective algebra $\mathcal{F}_0$ in V. It follows from 2.1.d that if $x \in \Lambda_V$ then (V, x) is a nilpotent $\mathcal{F}$-module. The variety Λ_V was introduced by Lusztig [Lus91, Section 12]. He also defined an open subset ${}^\delta \Lambda_V$ of Λ_V given by

$$(2.2.b) \qquad {}^\delta \Lambda_V = \{ x \in \Lambda_V \mid \dim \mathrm{Coker}(\bigoplus_{\substack{h \in H \\ \mathrm{In}(h)=i}} x_h) \leq \delta_i \text{ for any } i \in I \ \},$$

where $\delta \in \mathbb{Z}_{\geq 0}[I]$. The following proposition is proven in [Lus91, Section 12].

Proposition. *Let $v = \dim V$. Then*

2.2.c. ${}^\delta \Lambda_V$ *is either empty or has pure dimension* $\frac{1}{2} < Xv, v >$,

2.2.d. Λ_V *has pure dimension* $\frac{1}{2} < Xv, v >$.

2.3. **Stability and G_V-action.** Let $(x, p, q) \in \Lambda_{D,V}$ and E be a graded subspace of V. Then $\overline{E}$ (resp. $\underline{E}$) denotes the smallest $\mathcal{F}$-submodule of (V, x) containing E (resp. the largest $\mathcal{F}$-submodule of (V, x) contained in E).

Note that the triple $(x|_{q^{-1}(0)}, 0, 0)$ (resp. $(x|_{V/\overline{p(D)}}, 0, 0)$) belongs to $\Lambda_{\underline{q^{-1}(0)}}$ (resp. $\Lambda_{V/\overline{p(D)}}$). Nakajima [Nak94, Nak98] defined open (possibly empty) subsets of the variety $\Lambda_{D,V}$ given as sets of points satisfying some stability conditions. Two examples of such subsets are $\Lambda_{D,V}^s$ (the set of *stable* points) and $\Lambda_{D,V}^{*s}$ (the set of $*$-*stable* points). The set $\Lambda_{D,V}^s$ (resp. $\Lambda_{D,V}^{*s}$) is the set of all triples $(x, p, q) \in \Lambda_{D,V}$ such that $\overline{p(D)} = V$ (resp. $\underline{q^{-1}(0)} = 0$). Let $\Lambda_{D,V}^{s,*s} = \Lambda_{D,V}^s \cap \Lambda_{D,V}^{*s}$.

Let $G_V = \mathrm{Aut}_{\mathbb{C}^I} V = \prod_{i \in I} GL(V_i)$ be the group of graded automorphisms of V. The group G_V acts on Λ_V and $\Lambda_{D,V}$ as follows:

$$(g, x) \to x^g$$

$$(g, (x, p, q)) \to (x^g, p^g, q^g) \ ,$$

where $g \in G_V$, $x \in \Lambda_V$ (resp. $(x, p, q) \in \Lambda_{D,V}$), and

$$x_h^g = g_{\mathrm{In}(h)} x_h g_{\mathrm{Out}(h)}^{-1} \ , \qquad p_i^g = g_i p_i \ , \qquad q_i^g = q_i g_i^{-1} \ ,$$

for $h \in H$, $i \in I$.

The subsets $\Lambda_{D,V}^s$, $\Lambda_{D,V}^{*s}$, and $\Lambda_{D,V}^{s,*s}$, are preserved by the G_V-action.

The following proposition due to Nakajima and Crawley-Boevey explains the importance of the open sets $\Lambda_{D,V}^s$ and $\Lambda_{D,V}^{*s}$.

Proposition. *Let $d = \dim D$, $v = \dim V$. Then*

2.3.a. $\Lambda_{D,V}^s$ *is empty or an irreducible smooth variety of dimension*

$$\dim \Lambda_{D,V}^s = < Xv, v > + 2 < d, v > - < v, v > \ ;$$

2.3.b. $\Lambda_{D,V}^{*s}$ *is empty or an irreducible smooth variety of dimension*

$$\dim \Lambda_{D,V}^{*s} = <Xv, v> +2 <d, v> - <v, v> ;$$

2.3.c. $\Lambda_{D,V}^{s,*s}$ *is empty or an irreducible smooth variety of dimension*

$$\dim \Lambda_{D,V}^{s,*s} = <Xv, v> +2 <d, v> - <v, v> ;$$

2.3.d. $\Lambda_{D,V}^{s,*s}$ *is non-empty if and only if* $d - 2v + Xv \in \mathbb{Z}_{\geq 0}[I]$;
2.3.e. G_V-*action is free on* $\Lambda_{D,V}^s$, $\Lambda_{D,V}^{*s}$, *and* $\Lambda_{D,V}^{s,*s}$.

See (2.1.a) for the definition of the matrix X.

Proof. The smoothness part of 2.3.b and the statement 2.3.e in the case of $\Lambda_{D,V}^{*s}$ are proven by Nakajima [Nak98, Lemma 3.10]. The fact that $\Lambda_{D,V}^{*s}$ is connected is proven by Crawley-Boevey [CB00, Remarks after the Introduction]).

The proof of 2.3.a is analogous to the one of 2.3.b, or one can deduce the former from the latter by transposing (x, p, q). Either argument also proves the part of 2.3.e concerning $\Lambda_{D,V}^s$.

2.3.c (and the corresponding part of 2.3.e) follows from the definition of $\Lambda_{D,V}^{s,*s}$ ($\Lambda_{D,V}^{s,*s} = \Lambda_{D,V}^s \cap \Lambda_{D,V}^{*s}$), 2.3.a and 2.3.b.

2.3.d is proven by Nakajima [Nak98, Sections 10.5 – 10.9] (see also [Lus00a, Proposition 1.11]). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

It follows from 2.3.e and 2.3.a that $\mathfrak{M}_{D,V}^s = \Lambda_{D,V}^s / G_V$ is naturally an algebraic variety of dimension

$$\dim \mathfrak{M}_{D,V}^s = <Xv, v> +2 <d, v> -2 <v, v> ,$$

and the natural projection $\Lambda_{D,V}^s \to \mathfrak{M}_{D,V}^s$ is a principal G_V-bundle. The same holds for $\mathfrak{M}_{D,V}^{*s} = \Lambda_{D,V}^{*s} / G_V$ and $\mathfrak{M}_{D,V}^{s,*s} = \Lambda_{D,V}^{s,*s} / G_V$. The varieties $\mathfrak{M}_{D,V}^s$, $\mathfrak{M}_{D,V}^{*s}$, and $\mathfrak{M}_{D,V}^{s,*s}$ are called *quiver varieties* (cf. [Nak94, Nak98]).

Since $\mathfrak{M}_{D,V}^s$, $\mathfrak{M}_{D,V}^{*s}$, and $\mathfrak{M}_{D,V}^{s,*s}$, are G_V-orbit spaces, they do not depend on V as long as $\dim V = v$ is fixed. Also if $\dim D = \dim D'$ the above varieties are isomorphic (non canonically) to the corresponding varieties defined using D' instead of D. As D never vary, the following notation is often used (cf. [Nak94, Nak98]): $\mathfrak{M}^s(d, v) = \mathfrak{M}_{D,V}^s$, $\mathfrak{M}^{*s}(d, v) = \mathfrak{M}_{D,V}^{*s}$, $\mathfrak{M}^{s,*s}(d, v) = \mathfrak{M}_{D,V}^{s,*s}$.

2.4. **The variety** $\mathfrak{M}^s(d, v_0, v)$. Lusztig [Lus00a, Section 1] introduced two kinds of subsets of $\Lambda_{D,V}$. Namely, given a $\mathbb{Z}[I]$-graded subspace $U \subset V$ let

$$\Lambda_{D,V,U} = \{(x, p, q) \in \Lambda_{D,V} \mid \underline{q^{-1}(0)} = U\} ,$$

and given $v_0 \in \mathbb{Z}_{\geq 0}[I]$ let

$$\Lambda_{D,V,v_0} = \{(x, p, q) \in \Lambda_{D,V} \mid \dim \underline{q^{-1}(0)} = v - v_0\} .$$

The following proposition is proven in [Lus00a, 1.8].

Proposition. $\Lambda_{D,V,U}$ *and* Λ_{D,V,v_0} *are locally closed in* $\Lambda_{D,V}$.

Let $\Lambda_{D,V,U}^s = \Lambda_{D,V,U} \cap \Lambda_{D,V}^s$, $\Lambda_{D,V,v_0}^s = \Lambda_{D,V,v_0} \cap \Lambda_{D,V}^s$, and

$$\mathfrak{M}^s(d, v_0, v) = \Lambda_{D,V,v_0}^s / G_V .$$

2.5. **The set** $\mathcal{M}(d, v_0, v)$**.** Let U, T be complimentary $\mathbb{Z}[I]$-graded subspaces of V. Then one has a natural map (cf. [Lus00a])

$$\gamma : \Lambda_{D,V,U} \to \Lambda_U \times \Lambda_{D,T}^{*s} ,$$

given by

$$\gamma((x,p,q)) = (x^{UU}, (x^{TT}, p^{TD}, q^{DT})) ,$$

where the notation p^{TD} means the $D \to T$ component of the block-matrix $p : D \to U \oplus T$, and, similarly, x^{UU}, x^{TT}, q^{DT} are block-components of x and q. The map γ is well-defined because for $(x, p, q) \in \Lambda_{D,V,U}$, the subspace U is equal to $\underline{q^{-1}(0)}$ (i.e. it is the maximal $\mathcal{F}$-submodule contained in the kernel of q). The following proposition is due to Lusztig [Lus00a, Section 1].

Proposition. *Let* $d = \dim D$, $v = \dim V$, $u = \dim U$, $t = \dim T = v - u$. *Then*

2.5.a. *the map* γ *is a vector bundle with fibers of dimension* $< d, u > + < Xt, u > - < t, u >$;

2.5.b. *if* $\Lambda_{D,V,U}^s$ *is non-empty then* $\gamma(\Lambda_{D,V,U}^s) = {}^\delta\Lambda_U \times \Lambda_{D,T}^{s,*s}$, *where* $\delta = d - 2t + Xt$ *(the matrix* X *is defined in (2.1.a)), and* $\Lambda_{D,V,U}^s$ *is an open dense subset in* $\gamma^{-1}({}^\delta\Lambda_U \times \Lambda_{D,T}^{s,*s})$;

2.5.c. $\Lambda_{D,V,U}^s$ *is empty or has pure dimension*

$$\dim \Lambda_{D,V,U}^s = \frac{1}{2} < Xu, u > + < Xt, v > + < d, v > + < d, t > - < t, v > ;$$

2.5.d. Λ_{D,V,v_0}^s *is empty or has pure dimension*

$$\dim \Lambda_{D,V,v_0}^s = \frac{1}{2} < Xv, v > + \frac{1}{2} < Xv_0, v_0 > + < d, v > + < d, v_0 > - < v_0, v_0 > ;$$

2.5.e. $\mathfrak{M}^s(d, v_0, v)$ *is empty or has pure dimension*

$$\dim \mathfrak{M}^s(d, v_0, v) =$$
$$= \frac{1}{2} < Xv, v > + \frac{1}{2} < Xv_0, v_0 > + < d, v > + < d, v_0 > - < v_0, v_0 > - < v, v >$$
$$= \frac{1}{2} \dim \mathfrak{M}^s(d, v) + \frac{1}{2} \dim \mathfrak{M}^{s,*s}(d, v_0) .$$

Proof. 2.5.a and 2.5.b are proven in [Lus00a, Proposition 1.16] (see the proof of Proposition 2.7 for a similar argument).

2.5.c follows from 2.5.a, 2.5.b, 2.3.c, and 2.2.d.

Let Gr_w^V denote the variety of all $\mathbb{Z}[I]$-graded subspaces of graded dimension w in V. It is a G_V-homogeneous variety with connected stabilizer of a point, and $\dim Gr_w^V = < w, v - w >$. The map $\Lambda_{D,V,v_0}^s \to Gr_{v-v_0}^V$ given by $(x, p, q) \to \underline{q^{-1}(0)}$ is a locally trivial fibration over $Gr_{v-v_0}^V$ with the fiber over $U \in Gr_{v-v_0}^V$ equal to $\Lambda_{D,V,U}^s$. Now 2.5.d follows from 2.5.c.

2.5.e follows from 2.5.d and 2.3.e.   □

The statement 2.5.e also follows from [Nak98, Proof of Theorem 7.2].

Let $\mathcal{M}(d, v_0, v)$ be the set of irreducible components of the variety $\mathfrak{M}^s(d, v_0, v)$. Since Λ_{D,V,v_0}^s is the total space of a principal G_V-bundle over $\mathfrak{M}^s(d, v_0, v)$, while $\Lambda_{D,V,U}^s$ (where $\dim U = v - v_0$) is a fiber of the fibration $\Lambda_{D,V,v_0}^s \to Gr_{v-v_0}^V$ over a simply connected homogeneous space, one has natural bijections between the sets of irreducible components of Λ_{D,V,v_0}^s and $\Lambda_{D,V,U}^s$ and the set $\mathcal{M}(d, v_0, v)$. Abusing

notation the former two sets (of irreducible components) will be also denoted by $\mathcal{M}(d, v_0, v)$.

Let $\mathcal{M}(d, v_0) = \bigsqcup_{v \in \mathbb{Z}[I]} \mathcal{M}(d, v_0, v)$. In 3.8 the set $\mathcal{M}(d, v_0)$ is equipped with a structure of $\mathfrak{g}$-crystal, which is shown (in 3.10) to coincide with the crystal of the canonical basis of a highest weight representation of $\mathfrak{g}$.

2.6. **Tensor product varieties and multiplicity varieties.** Given an n-tuple $\mathbf{v}$ of elements of $\mathbb{Z}[I]$, $\mathbf{v}^k$ denotes the k-th component of $\mathbf{v}$. Thus $\mathbf{v} = (\mathbf{v}^1, \dots, \mathbf{v}^n)$, $\mathbf{v}^k \in \mathbb{Z}[I]$ $(1 \le k \le n)$, $\mathbf{v}_i^k \in \mathbb{Z}$ $(i \in I)$.

Let D be a $\mathbb{Z}[I]$-graded $\mathbb{C}$-linear space, and $\mathbf{D} = \{0 = \mathbf{D}^0 \subset \mathbf{D}^1 \subset \mathbf{D}^2 \subset \dots \subset \mathbf{D}^n = D\}$ be a partial $\mathbb{Z}[I]$-graded flag in D. Then $\dim \mathbf{D}$ denotes an n-tuple of elements of $\mathbb{Z}[I]$ given as $(\dim \mathbf{D})^k = \dim \mathbf{D}^k - \dim \mathbf{D}^{k-1}$ (for $1 \le k \le n$).

Recall that if $(x, p, q) \in \Lambda_{D,V}$ and E is a $\mathbb{Z}[I]$-graded subspace of V then $\overline{E}$ (resp. $\underline{E}$) denotes the smallest $\mathcal{F}$-submodule of (V, x) containing E (resp. the largest $\mathcal{F}$-submodule of (V, x) contained in E).

Let D, V be $\mathbb{Z}[I]$-graded $\mathbb{C}$-linear spaces, $\mathbf{D} = \{0 = \mathbf{D}^0 \subset \mathbf{D}^1 \subset \mathbf{D}^2 \subset \dots \subset \mathbf{D}^n = D\}$ be a partial $\mathbb{Z}[I]$-graded flag in D. Then ${}^n\Pi_{D,\mathbf{D},V}^s$ denotes the set of all $(x, p, q) \in \Lambda_{D,V}^s$ such that

$$\text{(2.6.a)} \qquad \overline{p(\mathbf{D}^k)} \subset \underline{q^{-1}(\mathbf{D}^k)}$$

for all $k = 1, \dots, n$.

Note that if $(x, p, q) \in {}^n\Pi_{D,\mathbf{D},V}^s$ then $\overline{p(D)} = V$ (because $(x, p, q) \in \Lambda_{D,V}^s$), and the triple

$$\left(x|_{\overline{p(\mathbf{D}^k)}/(\overline{p(\mathbf{D}^k)} \cap \underline{q^{-1}(\mathbf{D}^{k-1})})} \,,\; p|_{\mathbf{D}^k} \mod \underline{q^{-1}(\mathbf{D}^{k-1})} \,,\; q|_{\overline{p(\mathbf{D}^k)}} \mod \mathbf{D}^{k-1} \right)$$

belongs to $\Lambda_{\mathbf{D}^k/\mathbf{D}^{k-1} ,\, \overline{p(\mathbf{D}^k)}/(\overline{p(\mathbf{D}^k)} \cap \underline{q^{-1}(\mathbf{D}^{k-1})})}^{s,*s}$ for all $k = 1, \dots, n$.

Let $\mathbf{v}$ be an n-tuple of elements of $\mathbb{Z}_{\ge 0}[I]$. Then

$${}^n\Pi_{D,\mathbf{D},V,\mathbf{v}}^s = \left\{ (x,p,q) \in {}^n\Pi_{D,\mathbf{D},V}^s \mid \dim\left(\overline{p(\mathbf{D}^k)} / (\overline{p(\mathbf{D}^k)} \cap \underline{q^{-1}(\mathbf{D}^{k-1})}) \right) = \mathbf{v}^k \right\}.$$

Let $\tilde{\mathbf{v}}$ be another n-tuple of elements of $\mathbb{Z}_{\ge 0}[I]$ such that $\sum_{k=1}^n \mathbf{v}^k + \sum_{k=1}^n \tilde{\mathbf{v}}^k = \dim V$. Then

$${}^n\Pi_{D,\mathbf{D},V,\mathbf{v},\tilde{\mathbf{v}}}^s =$$
$$= \left\{ (x,p,q) \in {}^n\Pi_{D,\mathbf{D},V,\mathbf{v}}^s \mid \dim\left((\overline{p(\mathbf{D}^k)} \cap \underline{q^{-1}(\mathbf{D}^{k-1})}) / \overline{p(\mathbf{D}^{k-1})} \right) = \tilde{\mathbf{v}}^k \right\}.$$

Let $\mathbf{V} = (0 = \mathbf{V}^0 \subset \tilde{\mathbf{V}}^1 \subset \mathbf{V}^1 \subset \tilde{\mathbf{V}}^2 \subset \dots \subset \tilde{\mathbf{V}}^n \subset \mathbf{V}^n = V)$ be a $2n$-step $\mathbb{Z}[I]$-graded partial flag in V. Then

$${}^n\Pi_{D,\mathbf{D},V,\mathbf{V}}^s = \left\{ (x,p,q) \in {}^n\Pi_{D,\mathbf{D},V}^s \mid \overline{p(\mathbf{D}^k)} = \mathbf{V}^k \,,\; \underline{q^{-1}(\mathbf{D}^{k-1})} \cap \overline{p(\mathbf{D}^k)} = \tilde{\mathbf{V}}^k \right\}.$$

Note that ${}^n\Pi_{D,\mathbf{D},V,\mathbf{V}}^s \subset {}^n\Pi_{D,\mathbf{D},V,\mathbf{v},\tilde{\mathbf{v}}}^s$, where $\mathbf{v}^k = \dim \mathbf{V}^k - \dim \tilde{\mathbf{V}}^k$, $\tilde{\mathbf{v}}^k = \dim \tilde{\mathbf{V}}^k - \dim \mathbf{V}^{k-1}$.

The proof of the following proposition is analogous to the proof of Proposition 2.4.

Proposition. *The sets* ${}^n\Pi_{D,\mathbf{D},V,\mathbf{v}}^s$, ${}^n\Pi_{D,\mathbf{D},V,\mathbf{v},\tilde{\mathbf{v}}}^s$, *and* ${}^n\Pi_{D,\mathbf{D},V,\mathbf{V}}^s$, *are locally closed in* $\Lambda_{D,V}$.

Let $^n\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{v}} = {}^n\Pi^s_{D,\mathbf{D},V,\mathbf{v}} \cap \Lambda^{s,*s}_{D,V}$ (a locally closed subset of $\Lambda_{D,V}$). The sets $^n\Pi^s_{D,\mathbf{D},V,\mathbf{v}}$ and $^n\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{v}}$ are invariant under the action of G_V, and, moreover, this action is free (because of 2.3.e). Hence $^n\mathfrak{T}_{D,\mathbf{D},V,\mathbf{v}} = {}^n\Pi^s_{D,\mathbf{D},V,\mathbf{v}}/G_V$ and $^n\mathfrak{S}_{D,\mathbf{D},V,\mathbf{v}} = {}^n\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{v}}/G_V$ are naturally quasi-projective varieties.

The variety $^n\mathfrak{T}_{D,\mathbf{D},V,\mathbf{v}}$ is called *tensor product variety* because of its role in the geometric description of the tensor product of finite dimensional representations of $\mathfrak{g}$. In particular it is shown below (cf. 3.10.c) that the set of irreducible components of $^n\mathfrak{T}_{D,\mathbf{D},V,\mathbf{v}}$ is in a bijection with a weight subset of the crystal of the canonical basis of a product of n representations of $\mathfrak{g}$.

The variety $^n\mathfrak{S}_{D,\mathbf{D},V,\mathbf{v}}$ is called *multiplicity variety* because as proven below (cf. 3.10.b) the number of its irreducible components is equal to the multiplicity of a certain representation of $\mathfrak{g}$ in a tensor product of n representations.

The varieties $^n\mathfrak{T}_{D,\mathbf{D},V,\mathbf{v}}$ and $^n\mathfrak{S}_{D,\mathbf{D},V,\mathbf{v}}$ do not depend on the choice of V as long as $\dim V = v$ is fixed, and they do not depend up to a non canonical isomorphism on D and $\mathbf{D}$. As D and $\mathbf{D}$ never vary in this paper, a simpler notation is used: $^n\mathfrak{T}(d,\mathbf{d},v,\mathbf{v}) = {}^n\mathfrak{T}_{D,\mathbf{D},V,\mathbf{v}}$ and $^n\mathfrak{S}(d,\mathbf{d},v,\mathbf{v}) = {}^n\mathfrak{S}_{D,\mathbf{D},V,\mathbf{v}}$, where $d = \dim D$, $\mathbf{d} = \dim \mathbf{D}$ (cf. the end of subsection 2.3).

2.7. Inductive construction of tensor product varieties. Let $\mathbf{V} = (0 = \mathbf{V}^0 \subset \tilde{\mathbf{V}}^1 \subset \mathbf{V}^1 \subset \tilde{\mathbf{V}}^2 \subset \ldots \subset \tilde{\mathbf{V}}^n \subset \mathbf{V}^n = V)$ be a $2n$-step $\mathbb{Z}[I]$-graded partial flag in V, and $\mathbf{D} = (0 = \mathbf{D}^0 \subset \mathbf{D}^1 \subset \ldots \subset \mathbf{D}^n = D)$ be an n-step $\mathbb{Z}[I]$-graded partial flag in D. Assume $n \geq 2$ and choose an integer k such that $0 < k < n$. Let U (resp. C) be a $\mathbb{Z}[I]$-graded subspace in V (resp. in D) complimentary to $\mathbf{V}^k$ (resp. $\mathbf{D}^k$). Let $\mathbf{V}' = (0 = \mathbf{V}^0 \subset \tilde{\mathbf{V}}^1 \subset \mathbf{V}^1 \subset \tilde{\mathbf{V}}^2 \subset \ldots \subset \tilde{\mathbf{V}}^k \subset \mathbf{V}^k)$ (resp. $\mathbf{D}' = (0 = \mathbf{D}^0 \subset \mathbf{D}^1 \subset \ldots \subset \mathbf{D}^k)$) be a $2k$-step $\mathbb{Z}[I]$-graded partial flag in $\mathbf{V}^k$ (resp. a k-step $\mathbb{Z}[I]$-graded partial flag in $\mathbf{D}^k$), obtained by restricting the flag $\mathbf{V}$ (resp. $\mathbf{D}$). Similarly, let $\mathbf{U} = (0 = \mathbf{U}^0 \subset \tilde{\mathbf{U}}^1 \subset \mathbf{U}^1 \subset \tilde{\mathbf{U}}^2 \subset \ldots \subset \tilde{\mathbf{U}}^{n-k} \subset \mathbf{U}^{n-k} = U)$ (resp. $\mathbf{C} = (0 = \mathbf{D}^0 \subset \mathbf{D}^1 \subset \ldots \subset \mathbf{D}^{n-k} = D)$) be a $2(n-k)$-step $\mathbb{Z}[I]$-graded partial flag in U (resp. an $(n-k)$-step $\mathbb{Z}[I]$-graded partial flag in C), obtained by taking intersections of the subspaces of the flag $\mathbf{V}$ (resp. $\mathbf{D}$) with U (resp. C).

One has a regular map

$$\rho_2 : {}^n\Pi^s_{D,\mathbf{D},V,\mathbf{V}} \to {}^k\Pi^s_{\mathbf{D}^k,\mathbf{D}',\mathbf{V}^k,\mathbf{V}'} \times {}^{n-k}\Pi^s_{C,\mathbf{C},U,\mathbf{U}}$$

given by

$$\rho_2(x,p,q) = ((x^{\mathbf{V}^k\mathbf{V}^k}, p^{\mathbf{V}^k\mathbf{D}^k}, q^{\mathbf{D}^k\mathbf{V}^k}), (x^{UU}, p^{UC}, q^{CU})),$$

where for example p^{UC} is the $C \to U$ component of the block-matrix $p : D = \mathbf{D}^k \oplus C \to \mathbf{V}^k \oplus U = V$, and the other maps in the RHS are defined similarly.

Proposition. *The map ρ_2 is a vector bundle with fibers of dimension*

$$< Xu, v-u > + < c, v-u > + < d-c, u > - < u, v-u >,$$

where $v = \dim V$, $d = \dim D$, $u = \dim U$, $c = \dim C$.

Proof. The fiber of the map ρ_2 over a point $((x^{\mathbf{V}^k\mathbf{V}^k}, p^{\mathbf{V}^k\mathbf{D}^k}, q^{\mathbf{D}^k\mathbf{V}^k}), (x^{UU}, p^{UC}, q^{CU}))$ in $^k\Pi^s_{\mathbf{D}^k,\mathbf{D}',\mathbf{V}^k,\mathbf{V}'} \times {}^{n-k}\Pi^s_{C,\mathbf{C},U,\mathbf{U}}$ consists of all linear maps

$$x^{\mathbf{V}^k U}_h \in \mathrm{Hom}_{\mathbb{C}}(U_{\mathrm{Out}(h)}, \mathbf{V}^k_{\mathrm{In}(h)}),$$

$$p^{\mathbf{V}^k C}_i \in \mathrm{Hom}_{\mathbb{C}}(C_i, \mathbf{V}^k_i),$$

$$q^{\mathbf{D}^k U}_i \in \mathrm{Hom}_{\mathbb{C}}(U_i, \mathbf{D}^k_i),$$

(where $i \in I$, $h \in H$) subject to the following linear equations

$$\sum_{\substack{h \in H \\ \mathrm{In}(h)=i}} \varepsilon(h)(x_h^{\mathbf{V}^k \mathbf{V}^k} x_h^{\mathbf{V}^k U} + x_h^{\mathbf{V}^k U} x_{\bar{h}}^{UU}) - p_i^{\mathbf{V}^k C} q_i^{CU} - p_i^{\mathbf{V}^k \mathbf{D}^k} q_i^{\mathbf{D}^k U} = 0$$

for any $i \in I$. Now the proposition follows from the fact that the linear map

(2.7.a)
$$(\oplus_{h \in H} \mathrm{Hom}_{\mathbb{C}}(U_{\mathrm{Out}(h)}, \mathbf{V}^k_{\mathrm{In}(h)})) \oplus (\oplus_{i \in I} \mathrm{Hom}_{\mathbb{C}}(C_i, \mathbf{V}^k_i)) \oplus (\oplus_{i \in I} \mathrm{Hom}_{\mathbb{C}}(U_i, \mathbf{D}^k_i)) \rightarrow$$
$$\rightarrow \oplus_{i \in I} \mathrm{Hom}_{\mathbb{C}}(U_i, \mathbf{V}^k_i)$$

given by

$$\{x^{\mathbf{V}^k U}, p^{\mathbf{V}^k C}, q^{\mathbf{D}^k U}\} \rightarrow \sum_{\substack{h \in H \\ \mathrm{In}(h)=i}} \varepsilon(h)(x_h^{\mathbf{V}^k \mathbf{V}^k} x_h^{\mathbf{V}^k U} + x_h^{\mathbf{V}^k U} x_{\bar{h}}^{UU}) - p_i^{\mathbf{V}^k C} q_i^{CU} - p_i^{\mathbf{V}^k \mathbf{D}^k} q_i^{\mathbf{D}^k U}$$

is surjective. To prove this let $a \in \oplus_{i \in I} \mathrm{Hom}_{\mathbb{C}}(\mathbf{V}^k_i, U_i)$ be orthogonal to the image of the above map with respect to the tr pairing, that is

$$\sum_{\substack{h \in H \\ \mathrm{In}(h)=i}} \varepsilon(h) \operatorname{tr}(x_h^{\mathbf{V}^k \mathbf{V}^k} x_{\bar{h}}^{\mathbf{V}^k U} a_i + x_h^{\mathbf{V}^k U} x_{\bar{h}}^{UU} a_i) - \operatorname{tr} p_i^{\mathbf{V}^k C} q_i^{CU} a_i - \operatorname{tr} p_i^{\mathbf{V}^k \mathbf{D}^k} q_i^{\mathbf{D}^k U} a_i = 0$$

for any $x^{\mathbf{V}^k U}$, $p^{\mathbf{V}^k C}$, $q^{\mathbf{D}^k U}$, and any $i \in I$. It follows that

$$a_{\mathrm{In}(h)} x_h^{\mathbf{V}^k \mathbf{V}^k} = x_h^{UU} a_{\mathrm{Out}(h)} \ ,$$
$$q_i^{CU} a_i = 0 \ ,$$
$$a_i p_i^{\mathbf{V}^k \mathbf{D}^k} = 0 \ ,$$

for any $h \in H$, $i \in I$. In particular the kernel of a contains the image of $p^{\mathbf{V}^k \mathbf{D}^k}$ and is $x^{\mathbf{V}^k \mathbf{V}^k}$-invariant. Since $(x^{\mathbf{V}^k \mathbf{V}^k}, p^{\mathbf{V}^k \mathbf{D}^k}, q^{\mathbf{D}^k \mathbf{V}^k})$ is stable it follows that $a = 0$ and hence the map (2.7.a) is surjective. The proposition is proven. $\square$

2.8. Dimensions of tensor product and multiplicity varieties. The following proposition gives the dimensions of various varieties defined above.

Proposition. *Let $d = \dim D$, $v = \dim V$. Then*

2.8.a. $^n\Pi^s_{D,\mathbf{D},V,\mathbf{V}}$ *is empty or has pure dimension*

$$\dim {}^n\Pi^s_{D,\mathbf{D},V,\mathbf{V}} = \frac{1}{2} <Xv, v> + <d, v> - \frac{1}{2} <v, v> +$$
$$+ \sum_{s=1}^{n} \left(\frac{1}{2} <X\mathbf{v}^s, \mathbf{v}^s> + <\mathbf{d}^s, \mathbf{v}^s> - \frac{1}{2} <\mathbf{v}^s, \mathbf{v}^s> + \frac{1}{2} <\tilde{\mathbf{v}}^s, \tilde{\mathbf{v}}^s> \right) ,$$

where $\mathbf{v}^k = \dim \mathbf{V}^k - \dim \tilde{\mathbf{V}}^k$, $\tilde{\mathbf{v}}^k = \dim \tilde{\mathbf{V}}^k - \dim \mathbf{V}^{k-1}$;

2.8.b. $^n\Pi^s_{D,\mathbf{D},V,\mathbf{v},\tilde{\mathbf{v}}}$ *is empty or has pure dimension*

$$\dim {}^n\Pi^s_{D,\mathbf{D},V,\mathbf{v},\tilde{\mathbf{v}}} = \frac{1}{2} <Xv, v> + <d, v> +$$
$$+ \sum_{s=1}^{n} \left(\frac{1}{2} <X\mathbf{v}^s, \mathbf{v}^s> + <\mathbf{d}^s, \mathbf{v}^s> - <\mathbf{v}^s, \mathbf{v}^s> \right) ;$$

2.8.c. $\, {}^n\Pi^s_{D,\mathbf{D},V,\mathbf{v}}$ *is empty or has pure dimension*

$$\dim {}^n\Pi^s_{D,\mathbf{D},V,\mathbf{v}} = \frac{1}{2} <Xv,v> + <d,v> +$$
$$+ \sum_{s=1}^{n}\left(\frac{1}{2}<X\mathbf{v}^s,\mathbf{v}^s> + <\mathbf{d}^s,\mathbf{v}^s> - <\mathbf{v}^s,\mathbf{v}^s>\right) ;$$

2.8.d. $\, {}^n\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{v}}$ *is empty or has pure dimension*

$$\dim {}^n\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{v}} = \frac{1}{2} <Xv,v> + <d,v> +$$
$$+ \sum_{s=1}^{n}\left(\frac{1}{2}<X\mathbf{v}^s,\mathbf{v}^s> + <\mathbf{d}^s,\mathbf{v}^s> - <\mathbf{v}^s,\mathbf{v}^s>\right) ;$$

2.8.e. *the tensor product variety* $\, {}^n\mathfrak{T}(d,\mathbf{d},v,\mathbf{v})$ *is empty or has pure dimension*

$$\dim {}^n\mathfrak{T}(d,\mathbf{d},v,\mathbf{v}) = \frac{1}{2}<Xv,v> + <d,v> - <v,v> +$$
$$+ \sum_{s=1}^{n}\left(\frac{1}{2}<X\mathbf{v}^s,\mathbf{v}^s> + <\mathbf{d}^s,\mathbf{v}^s> - <\mathbf{v}^s,\mathbf{v}^s>\right) =$$
$$= \frac{1}{2}\left(\dim \mathfrak{M}^s(d,v) + \sum_{s=1}^{n} \dim \mathfrak{M}^{s,*s}(\mathbf{d}^s,\mathbf{v}^s)\right) ;$$

2.8.f. *the multiplicity variety* $\, {}^n\mathfrak{S}(d,\mathbf{d},v,\mathbf{v})$ *is empty or has pure dimension*

$$\dim {}^n\mathfrak{S}(d,\mathbf{d},v,\mathbf{v}) = \frac{1}{2}<Xv,v> + <d,v> - <v,v> +$$
$$+ \sum_{s=1}^{n}\left(\frac{1}{2}<X\mathbf{v}^s,\mathbf{v}^s> + <\mathbf{d}^s,\mathbf{v}^s> - <\mathbf{v}^s,\mathbf{v}^s>\right) =$$
$$= \frac{1}{2}\left(\dim \mathfrak{M}^{s,*s}(d,v) + \sum_{s=1}^{n} \dim \mathfrak{M}^{s,*s}(\mathbf{d}^s,\mathbf{v}^s)\right) ;$$

Proof. 2.8.a follows by induction in n using Proposition 2.7. The base of the induction is provided by 2.5.c.

2.8.b follows from 2.8.a using the fact that $\, {}^n\Pi^s_{D,\mathbf{D},V,\mathbf{v},\tilde{\mathbf{v}}}$ is a fibration with fibers isomorphic to $\, {}^n\Pi^s_{D,\mathbf{D},V,\mathbf{V}}$ over the variety of graded $2n$-step partial flags in V with dimensions of the subfactors given by $\mathbf{v}$ and $\tilde{\mathbf{v}}$. The dimension of this flag variety is equal to

$$\frac{1}{2}\left(<v,v> - \sum_{s=1}^{n}(<\mathbf{v}^s,\mathbf{v}^s> + <\tilde{\mathbf{v}}^s,\tilde{\mathbf{v}}^s>)\right).$$

2.8.c follows from the fact that $\, {}^n\Pi^s_{D,\mathbf{D},V,\mathbf{v}}$ is a union of a finite number of locally closed subsets $\, {}^n\Pi^s_{D,\mathbf{D},V,\mathbf{v},\tilde{\mathbf{v}}}$ (for different $\tilde{\mathbf{v}}$) having identical dimensions (cf. 2.8.b).

2.8.d follows from the fact that $\, {}^n\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{v}}$ is open in $\, {}^n\Pi^s_{D,\mathbf{D},V,\mathbf{v}}$ and from 2.8.c.

To prove 2.8.e (resp. 2.8.f) note that $\, {}^n\mathfrak{T}(d,\mathbf{d},v,\mathbf{v}) = {}^n\Pi^s_{D,\mathbf{D},V,\mathbf{v}}/G_V$ (resp. $\, {}^n\mathfrak{S}(d,\mathbf{d},v,\mathbf{v}) = {}^n\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{v}}/G_V$) and the action of the group G_V on $\, {}^n\Pi^s_{D,\mathbf{D},V,\mathbf{v}}$ (resp. $\, {}^n\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{v}}$) is free. Now 2.8.e (resp. 2.8.f) follows from 2.8.c (resp. 2.8.d). $\square$

2.9. Irreducible components of tensor product varieties. Let ${}^{n}\mathcal{T}(d,\mathbf{d},v,\mathbf{v})$ (resp. ${}^{n}\mathcal{S}(d,\mathbf{d},v,\mathbf{v})$) denote the set of irreducible components of the tensor product variety ${}^{n}\mathfrak{T}(d,\mathbf{d},v,\mathbf{v})$ (resp. the multiplicity variety ${}^{n}\mathfrak{S}(d,\mathbf{d},v,\mathbf{v})$).

Since ${}^{n}\Pi_{D,\mathbf{D},V,\mathbf{v}}^{s}$ is the total space of a principal G_V-bundle over ${}^{n}\mathfrak{T}(d,\mathbf{d},v,\mathbf{v})$, the set of irreducible components of ${}^{n}\Pi_{D,\mathbf{D},V,\mathbf{v}}^{s}$ can be naturally identified with ${}^{n}\mathcal{T}(d,\mathbf{d},v,\mathbf{v})$. On the other hand, ${}^{n}\Pi_{D,\mathbf{D},V,\mathbf{v}}^{s}$ is a union of locally closed subsets ${}^{n}\Pi_{D,\mathbf{D},V,\mathbf{v},\tilde{\mathbf{v}}}^{s}$ (for different $\tilde{\mathbf{v}}$) having the same dimension (cf. Proposition 2.8). Hence

$$ {}^{n}\mathcal{T}(d,\mathbf{d},v,\mathbf{v}) = \bigsqcup_{\tilde{\mathbf{v}}} {}^{n}\mathcal{T}(d,\mathbf{d},v,\mathbf{v},\tilde{\mathbf{v}}) \ , $$

where $\tilde{\mathbf{v}}$ ranges over all n-tuples of elements of $\mathbb{Z}[I]$ such that $\sum_{s=1}^{n}\mathbf{v}^{s}+\sum_{s=1}^{n}\tilde{\mathbf{v}}^{s}=v$, and ${}^{n}\mathcal{T}(d,\mathbf{d},v,\mathbf{v},\tilde{\mathbf{v}})$ denotes the set of irreducible components of ${}^{n}\Pi_{D,\mathbf{D},V,\mathbf{v},\tilde{\mathbf{v}}}^{s}$. The variety ${}^{n}\Pi_{D,\mathbf{D},V,\mathbf{v},\tilde{\mathbf{v}}}^{s}$ is a locally trivial fibration over the (simply connected G_V-homogeneous) variety of all graded $2n$-step partial flags in V with dimensions of the subfactors given by $\mathbf{v}$ and $\tilde{\mathbf{v}}$ (as in 2.6). A fiber of this fibration is isomorphic to ${}^{n}\Pi_{D,\mathbf{D},V,\mathbf{V}}^{s}$, where $\mathbf{V}$ is a point of the base (a $2n$-step partial flag). It follows that the set of irreducible components of ${}^{n}\Pi_{D,\mathbf{D},V,\mathbf{V}}^{s}$ can be naturally identified with the set ${}^{n}\mathcal{T}(d,\mathbf{d},v,\mathbf{v},\tilde{\mathbf{v}})$. To summarize, ${}^{n}\mathcal{T}(d,\mathbf{d},v,\mathbf{v})$ is the set of irreducible components of ${}^{n}\Pi_{D,\mathbf{D},V,\mathbf{v}}^{s}$ and ${}^{n}\mathfrak{T}(d,\mathbf{d},v,\mathbf{v})$, ${}^{n}\mathcal{T}(d,\mathbf{d},v,\mathbf{v},\tilde{\mathbf{v}})$ is the set of irreducible components of ${}^{n}\Pi_{D,\mathbf{D},V,\mathbf{v},\tilde{\mathbf{v}}}^{s}$ and ${}^{n}\Pi_{D,\mathbf{D},V,\mathbf{V}}^{s}$, and ${}^{n}\mathcal{T}(d,\mathbf{d},v,\mathbf{v}) = \bigsqcup_{\tilde{\mathbf{v}}} {}^{n}\mathcal{T}(d,\mathbf{d},v,\mathbf{v},\tilde{\mathbf{v}})$.

Similarly ${}^{n}\mathcal{S}(d,\mathbf{d},v,\mathbf{v})$ is the set of irreducible components of ${}^{n}\mathfrak{S}(d,\mathbf{d},v,\mathbf{v})$ and ${}^{n}\Pi_{D,\mathbf{D},V,\mathbf{v}}^{s,*s}$.

Finally, let ${}^{n}\mathcal{T}(d,\mathbf{d},\mathbf{v}) = \bigsqcup_{v\in\mathbb{Z}_{\geq 0}[I]} {}^{n}\mathcal{T}(d,\mathbf{d},v,\mathbf{v})$.

2.10. The first bijection for a tensor product variety. Let $1 < k < n$. The vector bundle ρ_2 introduced in 2.7 induces a bijection of the sets of irreducible components

$$ \alpha_{k,n-k} : {}^{n}\mathcal{T}(d,\mathbf{d},v,\mathbf{v},\tilde{\mathbf{v}}) \overset{\sim}{\longrightarrow} {}^{k}\mathcal{T}(d',\mathbf{d}',v',\mathbf{v}',\tilde{\mathbf{v}}') \times {}^{n-k}\mathcal{T}(d'',\mathbf{d}'',v'',\mathbf{v}'',\tilde{\mathbf{v}}'') \ , $$

where $d,d',d'',v,v',v'' \in \mathbb{Z}_{\geq 0}[I]$, $d = d' + d''$, $v = v' + v''$, $\mathbf{d},\mathbf{v},\tilde{\mathbf{v}}$ are n-tuples of elements of $\mathbb{Z}_{\geq 0}[I]$, $\mathbf{d}',\mathbf{v}',\tilde{\mathbf{v}}'$ are k-tuples of elements of $\mathbb{Z}_{\geq 0}[I]$, $\mathbf{d}'',\mathbf{v}'',\tilde{\mathbf{v}}''$ are $(n-k)$-tuples of elements of $\mathbb{Z}_{\geq 0}[I]$, $\mathbf{d} = (\mathbf{d}',\mathbf{d}'')$, $\mathbf{v} = (\mathbf{v}',\mathbf{v}'')$, $\tilde{\mathbf{v}} = (\tilde{\mathbf{v}}',\tilde{\mathbf{v}}'')$, $v' = \sum_{s=1}^{k}(\mathbf{v}'^{s}+\tilde{\mathbf{v}}'^{s})$, $v'' = \sum_{s=1}^{n-k}(\mathbf{v}''^{s}+\tilde{\mathbf{v}}''^{s})$. Taking union over $\tilde{\mathbf{v}}$ one obtains a bijection (denoted again by $\alpha_{k,n-k}$)

$$ \alpha_{k,n-k} : {}^{n}\mathcal{T}(d,\mathbf{d},v,\mathbf{v}) \overset{\sim}{\longrightarrow} \bigsqcup_{\substack{v',v''\in\mathbb{Z}_{\geq 0}[I] \\ v'+v''=v}} {}^{k}\mathcal{T}(d',\mathbf{d}',v',\mathbf{v}') \times {}^{n-k}\mathcal{T}(d'',\mathbf{d}'',v'',\mathbf{v}'') \ . $$

Generalizing the map ρ_2 (cf. 2.7) one can consider a map (given by restriction of x, p, and q)

$$ \rho_3 : \quad {}^{n}\Pi_{D,\mathbf{D},V,\mathbf{V}}^{s} \rightarrow $$
$$ \rightarrow {}^{k}\Pi_{\mathbf{D}^{k},\mathbf{D}',\mathbf{V}^{k},\mathbf{V}'}^{s} \times {}^{l-k}\Pi_{\mathbf{D}^{l}/\mathbf{D}^{k},\mathbf{D}'',\mathbf{V}^{l}/\mathbf{V}^{k},\mathbf{V}''}^{s} \times {}^{n-l}\Pi_{D/\mathbf{D}^{l},\mathbf{D}''',V/\mathbf{V}^{l},\mathbf{V}'''}^{s} \ , $$

where $1 < k < l < n$, $\mathbf{D}'$ (resp. $\mathbf{D}''$, $\mathbf{D}'''$) is the k-step partial flag in $\mathbf{D}^{k}$ (resp. $(l-k)$-step partial flag in $\mathbf{D}^{l}/\mathbf{D}^{k}$, $(n-l)$-step partial flag in $D/\mathbf{D}^{l}$) induced by the n-step flag $\mathbf{D}$ in D, and, similarly, $\mathbf{V}'$ (resp. $\mathbf{V}''$, $\mathbf{V}'''$) is the $2k$-step partial flag in $\mathbf{V}^{k}$ (resp. $2(l-k)$-step partial flag in $\mathbf{V}^{l}/\mathbf{V}^{k}$, $2(n-l)$-step partial flag in $V/\mathbf{V}^{l}$) induced by the $2n$-step flag $\mathbf{V}$ in V.

The map ρ_3 can be represented in two ways as a composition of two maps ρ_2:

(2.10.a)

$$
\begin{array}{ccc}
{}^n\Pi^s_{D,\mathbf{D},V,\mathbf{V}} & \xrightarrow{\ \rho_2\ } & {}^l\Pi^s_{\mathbf{D}^l,\bar{\bar{\mathbf{D}}},V^l,\bar{\bar{\mathbf{V}}}} \times {}^{n-l}\Pi^s_{D/\mathbf{D}^l,\mathbf{D}''',V/\mathbf{V}^l,\mathbf{V}'''} \\[2mm]
\Big\downarrow{\scriptstyle\rho_2} & {\scriptstyle\rho_3}\searrow & \Big\downarrow{\scriptstyle\rho_2\times\mathrm{Id}} \\[2mm]
{}^k\Pi^s_{\mathbf{D}^k,\mathbf{D}',\mathbf{V}^k,\mathbf{V}'} \times {}^{n-k}\Pi^s_{D/\mathbf{D}^k,\bar{\mathbf{D}},V/\mathbf{V}^k,\bar{\mathbf{V}}} & \xrightarrow{\mathrm{Id}\times\rho_2} & {}^k\Pi^s_{\mathbf{D}^k,\mathbf{D}',\mathbf{V}^k,\mathbf{V}'}\times{}^{l-k}\Pi^s_{\mathbf{D}^l/\mathbf{D}^k,\mathbf{D}'',\mathbf{V}^l/\mathbf{V}^k,\mathbf{V}''}\times{}^{n-l}\Pi^s_{D/\mathbf{D}^l,\mathbf{D}''',V/\mathbf{V}^l,\mathbf{V}'''}
\end{array}
$$

where $\bar{\mathbf{D}}$ (resp. $\bar{\bar{\mathbf{D}}}$) is the partial flag in $D/\mathbf{D}^k$ (resp. partial flag in $\mathbf{D}^l$) induced by the flag $\mathbf{D}$ in D, and, similarly, $\bar{\mathbf{V}}$ (resp. $\bar{\bar{\mathbf{V}}}$) is the partial flag in $V/\mathbf{V}^k$ (resp. partial flag in $\mathbf{V}^l$) induced by the flag $\mathbf{V}$ in V. It follows from the commutativity of the diagram (2.10.a) and Proposition 2.7 that ρ_3 is a locally trivial fibration with a smooth connected fiber, and therefore, induces a bijection

$$\alpha_{k,l-k,n-l}:\quad {}^n\mathcal{T}((d'+d''+d'''),(\mathbf{d}',\mathbf{d}'',\mathbf{d}'''),v,(\mathbf{v}',\mathbf{v}'',\mathbf{v}'''))\xrightarrow{\sim}$$

$$\xrightarrow{\sim}\bigsqcup_{\substack{v',v'',v'''\in\mathbb{Z}_{\geq0}[I]\\v'+v''+v'''=v}}{}^k\mathcal{T}(d',\mathbf{d}',v',\mathbf{v}')\times{}^{l-k}\mathcal{T}(d'',\mathbf{d}'',v'',\mathbf{v}'')\times{}^{n-l}\mathcal{T}(d''',\mathbf{d}''',v''',\mathbf{v}''')$$

Moreover, the commutativity of the diagram (2.10.a) implies that

$$\alpha_{k,l-k,n-l}=(\alpha_{k,l-k}\times\mathrm{Id})\circ\alpha_{l,n-l}=(\mathrm{Id}\times\alpha_{l-k,n-k})\circ\alpha_{k,n-k}\ .$$

One can consider analogues of the fibrations ρ_2 and ρ_3 taking values in the product of any number $(\leq n)$ of varieties Π^s. These maps can be represented as compositions of the maps ρ_2 in several ways (cf. (2.10.a)), hence they are fibrations with smooth connected fibers and induce isomorphisms of the sets of irreducible components. The most important case is the map ρ_n that takes value in the product of n varieties ${}^1\Pi^s$ (recall that a variety ${}^1\Pi^s$ is the same as the variety Λ^s). More precisely, let $\hat{\mathbf{D}}^s$ (resp. $\hat{\mathbf{V}}^s$) be a $\mathbb{Z}[I]$-graded subspace in $\mathbf{D}^s$ complimentary to $\mathbf{D}^{s-1}$ (resp. a $\mathbb{Z}[I]$-graded subspace in $\mathbf{V}^s$ complimentary to $\mathbf{V}^{s-1}$) and let

$$\rho_n:\quad {}^n\Pi^s_{D,\mathbf{D},V,\mathbf{V}}\to\Lambda^s_{\hat{\mathbf{D}}^1,\hat{\mathbf{V}}^1}\times\ldots\times\Lambda^s_{\hat{\mathbf{D}}^n,\hat{\mathbf{V}}^n}$$

be a regular map given by

$$\rho_n(x,p,q)=((x^{\hat{\mathbf{V}}^1\hat{\mathbf{V}}^1},p^{\hat{\mathbf{V}}^1\hat{\mathbf{D}}^1},q^{\hat{\mathbf{D}}^1\hat{\mathbf{V}}^1}),\ldots,(x^{\hat{\mathbf{V}}^n\hat{\mathbf{V}}^n},p^{\hat{\mathbf{V}}^n\hat{\mathbf{D}}^n},q^{\hat{\mathbf{D}}^n\hat{\mathbf{V}}^n}))\ .$$

The map ρ_n can be represented (in many ways) as a composition of maps ρ_2. Therefore it is a fibration with smooth connected fibers and it induces a bijection

$$\alpha_n:\quad {}^n\mathcal{T}(d,\mathbf{d},v,\mathbf{v},\tilde{\mathbf{v}})\xrightarrow{\sim}\mathcal{M}(\mathbf{d}^1,\mathbf{v}^1,\mathbf{v}^1+\tilde{\mathbf{v}}^1)\times\ldots\times\mathcal{M}(\mathbf{d}^n,\mathbf{v}^n,\mathbf{v}^n+\tilde{\mathbf{v}}^n)$$

or, after taking union over $\tilde{\mathbf{v}}$,

$$\alpha_n:\quad {}^n\mathcal{T}(d,\mathbf{d},v,\mathbf{v})\xrightarrow{\sim}\bigsqcup_{\substack{\mathbf{u}^1,\ldots,\mathbf{u}^n\in\mathbb{Z}_{\geq0}[I]\\\mathbf{u}^1+\ldots+\mathbf{u}^n=v}}\mathcal{M}(\mathbf{d}^1,\mathbf{v}^1,\mathbf{u}^1)\times\ldots\times\mathcal{M}(\mathbf{d}^n,\mathbf{v}^n,\mathbf{u}^n)\ .$$

Finally, union over v gives a bijection

(2.10.b) $$\alpha_n:\quad {}^n\mathcal{T}(d,\mathbf{d},\mathbf{v})\xrightarrow{\sim}\mathcal{M}(\mathbf{d}^1,\mathbf{v}^1)\times\ldots\times\mathcal{M}(\mathbf{d}^n,\mathbf{v}^n)\ .$$

Recall that $\mathcal{M}(d, v, u)$ denotes the set of irreducible components of a certain locally closed subset of a quiver variety (cf. 2.5). The bijection α_n justifies the name "tensor product variety" given to $^n\mathfrak{T}(d, \mathbf{d}, v, \mathbf{v})$, since it is known that the set $\mathcal{M}(d, v)$ can be equipped with the structure of a $\mathfrak{g}$-crystal, which is isomorphic to the crystal of the canonical basis of an irreducible finite dimensional representation of $\mathfrak{g}$ (see 2.5 for references and 3.8 for a proof).

2.11. **The multiplicity variety for two multiples.** Let $D = D^1 \oplus D^2$, and $\mathbf{D} = (0 \subset D^1 \subset D)$ (a flag in D). Similarly, let $V = V^1 \oplus U \oplus V^2$, and $\mathbf{V} = (0 \subset V^1 \subset V^1 \oplus U \subset V)$. Then one has the following map

(2.11.a) $$\sigma_2 : {}^2\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{V}} \to \Lambda^{s,*s}_{D^1,V^1} \times \Lambda_U \times \Lambda^{s,*s}_{D^2,V^2} ,$$

given by

$$\sigma_2(x, p, q) = ((x^{V^1 V^1}, p^{V^1 D^1}, q^{D^1 V^1}), x^{UU}, (x^{V^2 V^2}, p^{V^2 D^2}, q^{D^2 V^2})) .$$

Proposition. *Let* $v^1 = \dim V^1$, $v^2 = \dim V^2$, $u = \dim U$, $d^1 = \dim D^1$, $d^2 = \dim D^2$, $v = \dim V = v^1 + v^2 + u$, $d = \dim D = d^1 + d^2$. *Then:*

2.11.b. *the image of* σ_2 *is* $\Lambda^{s,*s}_{D^1,V^1} \times \Lambda'_U \times \Lambda^{s,*s}_{D^2,V^2}$ *where* Λ'_U *is an open subset of* Λ_U,

2.11.c. σ_2 *is a fibration with smooth connected fibers of dimension*

$$\frac{1}{2}(<Xv, v> - <Xv^1, v^1> - <Xu, u> - <Xv^2, v^2>) +$$

$$+ <d, u> - <d^1, v^1> - <d^2, v^2> +$$

$$+ \frac{1}{2}(<v, v> - <v^1, v^1> - <u, u> - <v^2, v^2>) .$$

Proof. The proof is analogous to the proofs of Propositions 2.5 and 2.7. Namely given

$$((x^{V^1 V^1}, p^{V^1 D^1}, q^{D^1 V^1}), x^{UU}, (x^{V^2 V^2}, p^{V^2 D^2}, q^{D^2 V^2})) \in \Lambda^{s,*s}_{D^1,V^1} \times \Lambda_U \times \Lambda^{s,*s}_{D^2,V^2}$$

the fiber of σ_2 over this point is vector bundle σ' over an affine space consisting of all linear maps

$$x^{V^1 U}, x^{UV^2}, p^{UD^2}, q^{D^1 U}$$

subject to the equations

$$\sum_{\substack{h \in H \\ \text{In}(h)=i}} \varepsilon(h)(x_h^{V^1 V^1} x_h^{V^1 U} + x_h^{V^1 U} x_{\bar{h}}^{UU}) - p_i^{V^1 D^1} q_i^{D^1 U} = 0$$

$$\sum_{\substack{h \in H \\ \text{In}(h)=i}} \varepsilon(h)(x_h^{UU} x_h^{UV^2} + x_h^{UV^2} x_{\bar{h}}^{V^2 V^2}) - p_i^{UD^2} q_i^{D^2 V^2} = 0 .$$

Now 2.11.b follows from the condition that $(x, p, q) \in \Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{V}}$. The proof is similar to 2.5.b (cf. [Lus00a, Proposition 1.16]), or one can deduce 2.11.b from 2.5.b and its dual using the fact that $(x, p, q) \in \Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{V}}$ if and only if

$$(x^{(V^1 \oplus U)(V^1 \oplus U)}, p^{(V^1 \oplus U)D^1}, q^{D^1(V^1 \oplus U)}) \in \Lambda^{*s}_{D^1,V^1 \oplus U}$$

and

$$(x^{(U \oplus V^2)(U \oplus V^2)}, p^{(U \oplus V^2)D^2}, q^{D^2(U \oplus V^2)}) \in \Lambda^{s}_{D^2,U \oplus V^2} .$$

The fiber of σ' over a point

$$((x^{V^1V^1}, p^{V^1D^1}, q^{D^1V^1}), x^{UU}, (x^{V^2V^2}, p^{V^2D^2}, q^{D^2V^2})), (x^{V^1U}, x^{UV^2}, p^{UD^2}, q^{D^1U}))$$

is an affine space of all linear maps

$$x^{V^1V^2}, p^{V^1D^2}, q^{D^1V^2}$$

subject to the equations

$$\sum_{\substack{h \in H \\ \text{In}(h)=i}} \varepsilon(h)(x_h^{V^1V^1} x_{\bar{h}}^{V^1V^2} + x_h^{V^1V^2} x_{\bar{h}}^{V^2V^2}) - p_i^{V^1D^1} q_i^{D^1V^2} - p_i^{V^1D^2} q_i^{D^2V^2} = 0 \ .$$

One has to check that the systems of linear equations used in the proof are not overdetermined, which follows from the fact that $(x^{V^1V^1}, p^{V^1D^1}, q^{D^1V^1}) \in \Lambda_{D^1,V^1}^{s,*s}$, and $(x^{V^2V^2}, p^{V^2D^2}, q^{D^2V^2}) \in \Lambda_{D^2,V^2}^{s,*s}$ (cf. the proof of Proposition 2.7). $\qquad\square$

Note that because $\Lambda_{D,V}^{s,*s}$ is smooth and connected (cf. 2.3.c) the above Proposition implies that the set of irreducible components of ${}^2\Pi_{D,\mathbf{D},V,\mathbf{V}}^{s,*s}$ is in a natural bijection with the set of irreducible components of Λ'_U.

2.12. **The first bijection for a multiplicity variety.** Let $1 < k < n$. Consider the variety ${}^n\Pi_{D,\mathbf{D},V,\mathbf{V}}^{s,*s}$. Let $D^1 = \mathbf{D}^k$, D^2 be a $\mathbb{Z}[I]$-graded subspace in D complimentary to $\mathbf{D}^k$. Similarly let $V^1 = \mathbf{V}^k$, U be a $\mathbb{Z}[I]$-graded subspace in $\tilde{\mathbf{V}}^{k+1}$ complimentary to $\mathbf{V}^k$, and V^2 be a $\mathbb{Z}[I]$-graded subspace in V complimentary to $\tilde{\mathbf{V}}^{k+1}$. Let $\bar{\mathbf{D}} = (0 \subset D^1 \subset D)$ (a subflag of $\mathbf{D}$), $\bar{\mathbf{V}} = (0 \subset V^1 \subset V^1 \oplus U \subset D)$ (a subflag of $\mathbf{V}$), $\mathbf{D}'$ (resp. $\mathbf{D}''$) be the k-step flag in D^1 (resp. the $(n-k)$-step flag in D^2) obtained by considering the first k subspaces of $\mathbf{D}$ (resp. by taking intersections of the last $n-k$ subspaces of $\mathbf{D}$ with D^2), and $\mathbf{V}'$ (resp. $\mathbf{V}''$) be the $2k$-step flag in V^1 (resp. the $2(n-k)$-step flag in V^2) obtained by considering the first $2k$ subspaces of $\mathbf{V}$ (resp. by taking intersections of the last $2(n-k)$ subspaces of $\mathbf{V}$ with V^2). Then ${}^n\Pi_{D,\mathbf{D},V,\mathbf{V}}^{s,*s} \subset {}^2\Pi_{D,\bar{\mathbf{D}},V,\bar{\mathbf{V}}}^{s,*s}$, ${}^k\Pi_{D^1,\mathbf{D}',V^1,\mathbf{V}'}^{s,*s} \subset \Lambda_{D^1,V^1}^{s,*s}$, ${}^{n-k}\Pi_{D^2,\mathbf{D}'',V^2,\mathbf{V}''}^{s,*s} \subset \Lambda_{D^2,V^2}^{s,*s}$ and the map σ_2 (cf. (2.11.a)) restricts to a fibration (with the same fibers as those of σ_2)

$$\sigma_{k,n-k}: \quad {}^n\Pi_{D,\mathbf{D},V,\mathbf{V}}^{s,*s} \to {}^k\Pi_{D^1,\mathbf{D}',V^1,\mathbf{V}'}^{s,*s} \times \Lambda'_U \times {}^{n-k}\Pi_{D^2,\mathbf{D}'',V^2,\mathbf{V}''}^{s,*s} \ .$$

This fibration having smooth connected fibers it induces a bijection

$$\eta_{k,n-k}: \quad {}^n\mathcal{S}(d'+d'', (\mathbf{d}', \mathbf{d}''), v, (\mathbf{v}', \mathbf{v}''), (\tilde{\mathbf{v}}', v-v'-v'', \tilde{\mathbf{v}}'')) \xrightarrow{\sim}$$

$$\xrightarrow{\sim} {}^k\mathcal{S}(d', \mathbf{d}', v', \mathbf{v}', \tilde{\mathbf{v}}') \times {}^{n-k}\mathcal{S}(d'', \mathbf{d}'', v'', \mathbf{v}'', \tilde{\mathbf{v}}'') \times$$

$$\times {}^2\mathcal{S}(d'+d'', (d', d''), v, (v', v''), (0, v-v'-v'')) \ ,$$

where the last multiple in the RHS represents the set of irreducible components of Λ', which is in a natural bijection (induced by the fibration σ_2 – cf. 2.11) with the set of irreducible components of ${}^2\Pi_{D,\bar{\mathbf{D}},V,\bar{\mathbf{V}}}^{s,*s}$.

Taking union over $\tilde{\mathbf{v}}'$, and $\tilde{\mathbf{v}}''$ one obtains a bijection

$$\eta_{k,n-k}: \quad {}^n\mathcal{S}(d'+d'', (\mathbf{d}', \mathbf{d}''), v, (\mathbf{v}', \mathbf{v}'')) \xrightarrow{\sim}$$

$$\xrightarrow{\sim} \bigsqcup_{v',v'' \in \mathbb{Z}_{\geq 0}[I]} {}^k\mathcal{S}(d', \mathbf{d}', v', \mathbf{v}') \times {}^{n-k}\mathcal{S}(d'', \mathbf{d}'', v'', \mathbf{v}'') \times$$

$$\times {}^2\mathcal{S}(d'+d'', (d', d''), v, (v', v'')) \ ,$$

which is an analogue of the recurrence relation between multiplicities in tensor products of representations of $\mathfrak{g}$.

As a generalization of the map $\sigma_{k,n-k}$ one can consider a map

$$(2.12.a) \quad \sigma_n : \quad {}^n\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{V}} \rightarrow$$
$$\rightarrow \Lambda^{s,*s}_{\mathbf{D}^1/\mathbf{D}^0,\mathbf{V}^1/\tilde{\mathbf{V}}^1} \times \ldots \times \Lambda^{s,*s}_{\mathbf{D}^n/\mathbf{D}^{n-1},\mathbf{V}^n/\tilde{\mathbf{V}}^n} \times \Lambda_{\tilde{\mathbf{V}}^1/\mathbf{V}^0} \times \ldots \times \Lambda_{\tilde{\mathbf{V}}^n/\mathbf{V}^{n-1}}$$

defined by restricting x, p, and q. The map σ_n can be represented (in many ways) as a composition of the maps of type $\sigma_{n,n-k}$ for different n, and k (cf. a similar statement for the map ρ_n in 2.10). Therefore the image of σ_n is equal to

$$\Lambda^{s,*s}_{\mathbf{D}^1/\mathbf{D}^0,\mathbf{V}^1/\tilde{\mathbf{V}}^1} \times \ldots \times \Lambda^{s,*s}_{\mathbf{D}^n/\mathbf{D}^{n-1},\mathbf{V}^n/\tilde{\mathbf{V}}^n} \times X \;,$$

where X is an open subset of $\Lambda_{\tilde{\mathbf{V}}^1/\mathbf{V}^0} \times \ldots \times \Lambda_{\tilde{\mathbf{V}}^n/\mathbf{V}^{n-1}}$, and ${}^n\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{V}}$ is an open subset of the total space of a locally-trivial fibration σ' over the image of σ with a smooth connected fiber, and such that σ' restricted to ${}^n\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{V}}$ is equal to σ. In particular the set ${}^n\mathcal{S}(d,\mathbf{d},v,\mathbf{v})$ of irreducible components of ${}^n\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{V}}$ is in a natural bijection with the set of irreducible components of X.

If $\sum_{k=1}^n \mathbf{v}^k = v$ the map σ_n is a vector bundle

$$\sigma_n : \quad {}^n\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{V}} \rightarrow \Lambda^{s,*s}_{\mathbf{D}^1/\mathbf{D}^0,\mathbf{V}^1/\mathbf{V}^0} \times \ldots \times \Lambda^{s,*s}_{\mathbf{D}^n/\mathbf{D}^{n-1},\mathbf{V}^n/\mathbf{V}^{n-1}} \;,$$

which implies the following proposition

Proposition. *Assume that $\sum_{k=1}^n \mathbf{v}^k = v$, and $\Lambda^{s,*s}_{\mathbf{D}^k/\mathbf{D}^{k-1},\mathbf{V}^k/\mathbf{V}^{k-1}}$ is non-empty for all $k = 1 \ldots n$. Then the set ${}^n\mathcal{S}(d,\mathbf{d},v,\mathbf{v})$ is a one-element set.*

2.13. **The second bijection.** Recall that the tensor product variety ${}^n\Pi^s_{D,\mathbf{D},V,\mathbf{v}}$ is a locally closed subset of $\Lambda^s_{D,V}$, and that $\Lambda^s_{D,V,U}$ denotes the locally closed subset of $\Lambda^s_{D,V}$ consisting of all $(x,p,q) \in \Lambda^s_{D,V}$ such that $\underline{q^{-1}(0) = U}$ (cf. 2.5). Let ${}^n\Pi^s_{D,\mathbf{D},V,U,\mathbf{v}} = {}^n\Pi^s_{D,\mathbf{D},V,\mathbf{v}} \cap \Lambda^s_{D,V,U}$. Let T be a $\mathbb{Z}[I]$-graded subspace of V complimentary to U. The vector bundle $\gamma : \Lambda_{D,V,U} \rightarrow \Lambda_U \times \Lambda^{*s}_{D,T}$ (cf. Proposition 2.5) restricted to ${}^n\Pi^s_{D,\mathbf{D},V,U,\mathbf{v}}$ has the image equal to ${}^\delta\Lambda_U \times {}^n\Pi^{s,*s}_{D,\mathbf{D},T,\mathbf{v}}$, where ${}^\delta\Lambda_U$ is as in Proposition 2.5.b, and moreover

$$\gamma^{-1}({}^\delta\Lambda_U \times {}^n\Pi^{s,*s}_{D,\mathbf{D},T,\mathbf{v}}) \cap \Lambda^s_{D,V,U} = {}^n\Pi^s_{D,\mathbf{D},V,U,\mathbf{v}} \;.$$

Hence ${}^n\Pi^s_{D,\mathbf{D},V,U,\mathbf{v}}$ is open in $\gamma^{-1}({}^\delta\Lambda_U \times {}^n\Pi^{s,*s}_{D,\mathbf{D},T,\mathbf{v}})$. Since fibers of γ have dimension $<d,u> - <u,v-u> + <Xu,v-u>$ (cf. Proposition 2.5), it follows that ${}^n\Pi^s_{D,\mathbf{D},V,U,\mathbf{v}}$ has pure dimension

$$\dim {}^n\Pi^s_{D,\mathbf{D},V,U,\mathbf{v}} =$$
$$= \dim {}^\delta\Lambda_U + \dim {}^n\Pi^{s,*s}_{D,\mathbf{D},T,\mathbf{v}} + <d,u> - <u,v-u> + <Xu,v-u> =$$
$$= \dim {}^n\Pi^s_{D,\mathbf{D},V,\mathbf{v}} - <u,v-u> \;,$$

and the set of irreducible components of ${}^n\Pi^s_{D,\mathbf{D},V,U,\mathbf{v}}$ is in a natural bijection (induced by the restriction of the vector bundle γ to ${}^n\Pi^s_{D,\mathbf{D},V,U,\mathbf{v}}$) with the set

$$\mathcal{M}(d,v-u,v) \times {}^n\mathcal{S}(d,\mathbf{d},v-u,\mathbf{v}) \;,$$

where $\mathcal{M}(d,v-u,v)$ represents the set of irreducible components of ${}^\delta\Lambda_U$, which is in a natural bijection (induced by γ – cf. 2.5) with the set of irreducible components of the quiver variety $\mathfrak{M}(d,v-u,v)$, and ${}^n\mathcal{S}(d,\mathbf{d},v-u,\mathbf{v})$ represents the

set of irreducible components of the variety ${}^n\Pi^{s,*s}_{D,\mathbf{D},T,\mathbf{v}}$ (or the multiplicity variety
${}^n\mathfrak{S}(d,\mathbf{d},v-u,\mathbf{v})$ – cf. 2.9).

Let ${}^n\Pi^s_{D,\mathbf{D},V,u,\mathbf{v}}$ be the set of all $(x,p,q) \in {}^n\Pi^s_{D,\mathbf{D},V,\mathbf{v}}$ such that $\dim \underline{q^{-1}(0)} = u$.
It is the total space of a fibration over the graded Grassmannian Gr^V_u with the fiber
over $U \in Gr^V_u$ equal to ${}^n\Pi^s_{D,\mathbf{D},V,U,\mathbf{v}}$. Hence ${}^n\Pi^s_{D,\mathbf{D},V,u,\mathbf{v}}$ has pure dimension

$$\dim {}^n\Pi^s_{D,\mathbf{D},V,u,\mathbf{v}} = \dim {}^n\Pi^s_{D,\mathbf{D},V,\mathbf{v}} \ .$$

In particular the dimension does not depend on u. Therefore one obtains the
following natural bijection of sets of irreducible components

$$(2.13.\mathrm{a}) \qquad \beta_n : \ {}^n\mathcal{T}(d,\mathbf{d},v,\mathbf{v}) \xrightarrow{\sim} \bigsqcup_{v_0 \in \mathbb{Z}_{\geq 0}} {}^n\mathcal{S}(d,\mathbf{d},v_0,\mathbf{v}) \times \mathcal{M}(d,v_0,v) \ .$$

The bijection β_n is an analogue of the direct sum decomposition for the ten-
sor product of n representations of $\mathfrak{g}$. In particular (2.13.a) explains the name
"multiplicity variety" given to ${}^n\mathfrak{S}(d,\mathbf{d},v_0,\mathbf{v})$.

2.14. The tensor decomposition bijection. Let τ_n be a bijection

$$(2.14.\mathrm{a}) \quad \tau_n : \bigsqcup_{\substack{\mathbf{u}^1,\ldots,\mathbf{u}^n \in \mathbb{Z}[I] \\ \mathbf{u}^1+\ldots+\mathbf{u}^n=v}} \mathcal{M}(\mathbf{d}^1,\mathbf{v}^1,\mathbf{u}^1) \times \ldots \times \mathcal{M}(\mathbf{d}^n,\mathbf{v}^n,\mathbf{u}^n) \xrightarrow{\sim}$$

$$\xrightarrow{\sim} \bigsqcup_{v_0 \in \mathbb{Z}_{\geq 0}} {}^n\mathcal{S}(d,\mathbf{d},v_0,\mathbf{v}) \times \mathcal{M}(d,v_0,v)$$

given by

$$\tau_n = \beta_n \circ \alpha_n^{-1} \ .$$

Taking union over v one obtains a bijection

$$(2.14.\mathrm{b}) \ \ \tau_n : \mathcal{M}(\mathbf{d}^1,\mathbf{v}^1) \times \ldots \times \mathcal{M}(\mathbf{d}^n,\mathbf{v}^n) \xrightarrow{\sim} \bigsqcup_{v_0 \in \mathbb{Z}_{\geq 0}} {}^n\mathcal{S}(d,\mathbf{d},v_0,\mathbf{v}) \times \mathcal{M}(d,v_0) \ .$$

It follows from the definitions of the bijections α_n (cf. 2.10), β_n (cf. 2.13), and
$\eta_{k,n-k}$ (cf. 2.12), that

$$(2.14.\mathrm{c}) \qquad\qquad \tau_n = (\eta_{k,n-k}^{-1} \times \tau_2) \circ (\tau_k \times \tau_{n-k}) \ .$$

More precisely, the following diagram of bijections is commutative:

(2.14.d)

$$\mathcal{M}(\mathbf{d}^1, \mathbf{v}^1) \times \ldots \times \mathcal{M}(\mathbf{d}^k, \mathbf{v}^k) \times$$
$$\times \mathcal{M}(\mathbf{d}^{k+1}, \mathbf{v}^{k+1}) \times \ldots \times \mathcal{M}(\mathbf{d}^n, \mathbf{v}^n) \quad \xrightarrow{\quad \tau_n \quad}$$

$$\downarrow \tau_k \times \tau_{n-k}$$

$$\bigsqcup_{v_0', v_0''} \qquad {}^k\mathcal{S}(d', \mathbf{d}', v_0', \mathbf{v}') \times \mathcal{M}(d', v_0') \times$$
$$\phantom{\bigsqcup_{v_0', v_0''}} \qquad {}^{n-k}\mathcal{S}(d'', \mathbf{d}'', v_0'', \mathbf{v}'') \times \mathcal{M}(d'', v_0'')$$

$$\downarrow \mathrm{Id} \times P_{23} \times \mathrm{Id}$$

$$\bigsqcup_{v_0', v_0''} \qquad {}^k\mathcal{S}(d', \mathbf{d}', v_0', \mathbf{v}') \times {}^{n-k}\mathcal{S}(d'', \mathbf{d}'', v_0'', \mathbf{v}'') \times$$
$$\phantom{\bigsqcup_{v_0', v_0''}} \qquad \mathcal{M}(d', v_0') \times \mathcal{M}(d'', v_0'')$$

$$\downarrow \mathrm{Id} \times \mathrm{Id} \times \tau_2$$

$$\bigsqcup_{v_0, v_0', v_0''} \qquad {}^k\mathcal{S}(d', \mathbf{d}', v_0', \mathbf{v}') \times {}^{n-k}\mathcal{S}(d'', \mathbf{d}'', v_0'', \mathbf{v}'') \times$$
$$\phantom{\bigsqcup_{v_0, v_0', v_0''}} \qquad {}^2\mathcal{S}(d' + d'', (d', d''), v_0, (v_0', v_0'')) \times \mathcal{M}(d, v_0)$$

$$\downarrow \eta_{k,n-k}^{-1} \times \mathrm{Id}$$

$$\bigsqcup_{v_0} \qquad {}^n\mathcal{S}(d' + d'', (\mathbf{d}', \mathbf{d}''), v_0, (\mathbf{v}', \mathbf{v}'')) \times \mathcal{M}(d, v_0) \longleftarrow$$

where $\mathbf{v}' = (\mathbf{v}^1, \ldots, \mathbf{v}^k)$, $\mathbf{v}'' = (\mathbf{v}^{k+1}, \ldots, \mathbf{v}^n)$, $\mathbf{d}' = (\mathbf{d}^1, \ldots, \mathbf{d}^k)$, and $\mathbf{d}'' = (\mathbf{d}^{k+1}, \ldots, \mathbf{d}^n)$.

In the next section it will be shown that the set $\mathcal{M}(d, v_0)$ can be equipped with a structure of a $\mathfrak{g}$-crystal, and that the bijection τ_n is a crystal isomorphism (if one replaces direct products of sets with tensor products of crystals and considers the set ${}^n\mathcal{S}$ as a trivial crystal).

3. Levi restriction and the crystal structure on quiver varieties

3.1. **A subquiver Q'.** Let $Q' = (I', H') \subset Q = (I, H)$ be a (full) subquiver of Q (i.e. the set I' of vertices of Q' is a subset of the set I of vertices of Q, and two vertices of Q' are connected by an oriented edge $h \in H'$ of Q' if and only if they are connected by an oriented edge of Q).

Let $H^{QQ'}$ (resp. $H^{Q'Q}$) denote the set of edges $h \in H$ of Q such that $\mathrm{Out}(h) \in Q'$ and $\mathrm{In}(h) \notin Q'$ (resp. $\mathrm{In}(h) \in Q'$ and $\mathrm{Out}(h) \notin Q'$).

In this section (until 3.9) subscripts Q and Q' are used to distinguish the varieties defined using the two quivers. For example, $_Q\mathfrak{M}^s_{D,V}$ is a quiver variety associated to Q (in particular, D and V are $\mathbb{Z}[I]$-graded $\mathbb{C}$-linear spaces), whereas $_{Q'}\mathfrak{M}^s_{D,V}$ is a quiver variety associated to Q' (and, correspondingly, D and V are $\mathbb{Z}[I']$-graded $\mathbb{C}$-linear spaces, or $\mathbb{Z}[I]$-graded spaces with zero components in degrees $i \in I\backslash I'$), $_Q\mathcal{F}$ is the path algebra of Q, whereas $_{Q'}\mathcal{F}$ is the path algebra of Q'. Note that $_{Q'}\mathcal{F}$ is a subalgebra of $_Q\mathcal{F}$.

3.2. **The set** $_{QQ'}\mathcal{M}(d, v_0, v)$. Let D and V be $\mathbb{Z}[I]$-graded $\mathbb{C}$-linear spaces with graded dimensions d and v respectively, and let D', V' be $\mathbb{Z}[I']$-graded $\mathbb{C}$-linear spaces defined as follows

$$D'_i = D_i \oplus (\oplus^{h \in H^{QQ'}}_{\mathrm{Out}(h)=i} V_{\mathrm{In}(h)}) \ ,$$
$$V'_i = V_i \ ,$$

for $i \in I' \subset I$. One has a regular map $\zeta_{QQ'} : {}_Q\Lambda_{D,V} \to {}_{Q'}\Lambda_{D',V'}$ given by

$$\zeta_{QQ'}((x, p, q)) = (x', p', q') \ ,$$

where

$$x'_h = x_h \text{ for } h \in H' \subset H \ ,$$
$$p'_i = \left(p_i, \{x_h\}^{h \in H^{Q'Q}}_{\mathrm{In}(h)=i}\right) \ ,$$
$$q'_i = \left(q_i, \{(\varepsilon(h))^{-1} x_h\}^{h \in H^{QQ'}}_{\mathrm{Out}(h)=i}\right) \ .$$

Let $_{QQ'}\Lambda^{*s}_{D,V}$ be the set of all $(x, p, q) \in {}_Q\Lambda_{D,V}$ such that $\zeta_{QQ'}((x, p, q)) \in {}_{Q'}\Lambda^{*s}_{D',V'}$. Let $_{QQ'}\Lambda^{s,*s}_{D,V} = {}_{QQ'}\Lambda^{*s}_{D,V} \cap {}_Q\Lambda^s_{D,V}$. Roughly speaking, $_{QQ'}\Lambda^{s,*s}_{D,V}$ is the open subset of $\Lambda_{D,V}$ consisting of all stable points that are also $*$-stable "at vertices of Q'".

Let $v_0 \in \mathbb{Z}[I]$ and let $_{QQ'}\Lambda^{s,*s}_{D,V,v_0} = {}_{QQ'}\Lambda^{s,*s}_{D,V} \cap {}_Q\Lambda^s_{D,V,v_0}$. Since $_{QQ'}\Lambda^{s,*s}_{D,V,v_0}$ is an open G_V-invariant subset of $_Q\Lambda^s_{D,V,v_0}$ one has the following proposition (cf. 2.5.d, 2.3.e).

Proposition. $_{QQ'}\Lambda^{s,*s}_{D,V,v_0}$ *is empty or has pure dimension*

$$\dim {}_{QQ'}\Lambda^{s,*s}_{D,V,v_0} =$$
$$= \frac{1}{2} <Xv, v> + \frac{1}{2} <Xv_0, v_0> + <d, v> + <d, v_0> - <v_0, v_0> \ ,$$

and G_V-action on $_{QQ'}\Lambda^{s,*s}_{D,V,v_0}$ is free.

It follows that $_{QQ'}\mathfrak{M}^{s,*s}(d, v_0, v) = {}_{QQ'}\Lambda^{s,*s}_{D,V,v_0}/G_V$ is naturally a quasi-projective variety of pure dimension

$$\dim {}_{QQ'}\mathfrak{M}^{s,*s}(d, v_0, v) = \frac{1}{2} <Xv, v> + \frac{1}{2} <Xv_0, v_0> +$$
$$+ <d, v> + <d, v_0> - <v_0, v_0> - <v, v> \ .$$

Let $_{QQ'}\mathcal{M}(d, v_0, v)$ be the set of irreducible components of $_{QQ'}\mathfrak{M}^{s,*s}(d, v_0, v)$. This set is also in a natural bijection (cf. the end of Section 2.5) with the sets of irreducible components of $_{QQ'}\Lambda^{s,*s}_{D,V,v_0}$, and of $_{QQ'}\Lambda^{s,*s}_{D,V,U} = {}_{QQ'}\Lambda^{s,*s}_{D,V} \cap {}_Q\Lambda^s_{D,V,U}$ (where $\dim U = v - v_0$).

3.3. **Levi restriction.** Given $(x, p, q) \in {}_Q\Lambda^s_{D,V}$ let ${}_{QQ'}\mathcal{K}(x, q)$ denote the maximal graded subspace $U \subset V$, satisfying the following conditions:

$$
\begin{array}{lll}
& U_i = \{0\} & \text{for any } i \notin I' , \\
(3.3.\text{a}) & x_h U \subset U & \text{for any } h \in H' , \\
& x_h U = 0 & \text{for any } h \in H^{QQ'} , \\
& q_i U = 0 & \text{for any } i \in I' .
\end{array}
$$

Let $w \in \mathbb{Z}[I'] \subset \mathbb{Z}[I]$, and W be a graded subspace of V with $\dim W \in \mathbb{Z}[I'] \subset \mathbb{Z}[I]$. Then ${}_{QQ'}\Lambda^s_{D,V,v_0,w}$ (resp. ${}_{QQ'}\Lambda^s_{D,V,v_0,W}$) denotes the set of all $(x, p, q) \in {}_Q\Lambda^s_{D,V,v_0}$ such that $\dim {}_{QQ'}\mathcal{K}(x, q) = w$ (resp. ${}_{QQ'}\mathcal{K}(x, q) = W$). Note that ${}_{QQ'}\Lambda^s_{D,V,v_0,w}$ is a fibration over a graded Grassmannian Gr^V_w with fibers isomorphic to ${}_{QQ'}\Lambda^s_{D,V,v_0,W}$, where $\dim W = w$.

Let W be as above and T be a graded subspace of V complimentary to W. Then one has a regular map (cf. 2.5)

$$
\nu_{QQ'} : \; {}_{QQ'}\Lambda^s_{D,V,v_0,W} \to {}_{Q'}\Lambda_W \times {}_{QQ'}\Lambda^{s,*s}_{D,T,v_0}
$$

given by

$$
\nu_{QQ'}((x, p, q)) = (x^{WW}, (x^{TT}, p^{TD}, q^{DT})) .
$$

The fiber of $\nu_{QQ'}$ over a point $(x, (y, p, q)) \in {}_{Q'}\Lambda_W \times {}_{QQ'}\Lambda^{s,*s}_{D,T,v_0}$ is the same as the fiber of the map γ (cf. 2.5) for the quiver Q' over the point $(x, \zeta_{QQ'}((y, p, q))) \in {}_{Q'}\Lambda_W \times {}_{Q'}\Lambda^{s,*s}_{D',T'}$, where $T'_i = T_i$ for $i \in I' \subset I$, and D' and $\zeta_{QQ'}$ are defined in 3.2. Therefore an analogue of the Proposition 2.5 holds and one obtains the following bijection between sets of irreducible components:

$$
\theta_{QQ'} : \; {}_Q\mathcal{M}(d, v_0, v) \xrightarrow{\sim}
$$
$$
\xrightarrow{\sim} \bigsqcup_{u \in \mathbb{Z}_{\geq 0}[I']} {}_{QQ'}\mathcal{M}(d, v_0, v - u) \times {}_{Q'}\mathcal{M}(\delta_{QQ'}(d, v), \rho_{QQ'}(v - u), \rho_{QQ'}(v))
$$

where

$$
\begin{array}{lll}
(\rho_{QQ'}(v))_i = v_i & \text{for} & v \in \mathbb{Z}[I] , \; i \in I' \subset I , \\
(\delta_{QQ'}(d, v))_i = d_i + \displaystyle\sum_{\substack{h \in H^{QQ'} \\ \text{Out}(h) = i}} v_{\text{In}(h)} & \text{for} & d, v \in \mathbb{Z}[I] , \; i \in I' \subset I .
\end{array}
$$

Union over v gives a bijection

$$
\theta_{QQ'} : \; {}_Q\mathcal{M}(d, v_0) \xrightarrow{\sim} \bigsqcup_{x \in \mathbb{Z}_{\geq 0}[I]} {}_{QQ'}\mathcal{M}(d, v_0, x) \times {}_{Q'}\mathcal{M}(\delta_{QQ'}(d, x), \rho_{QQ'}(x))
$$

The bijection $\theta_{QQ'}$ is an analogue of the restriction of a representation of $\mathfrak{g}$ to a Levi subalgebra of a parabolic subalgebra of $\mathfrak{g}$. This construction is a straightforward generalization of the reduction to $\mathfrak{sl}_2$ subalgebras introduced by Lusztig [Lus91, 12] and used by Nakajima [Nak98], Kashiwara and Saito [KS97].

3.4. **Levi restriction and the second bijection for tensor product varieties.** (cf. 2.13)

Let D, V be $\mathbb{Z}[I]$-graded $\mathbb{C}$-linear spaces, $\mathbf{D}$ be an n-step partial flag in D, and $\mathbf{v}$ be an n-tuple of elements of $\mathbb{Z}[I]$. Recall (cf. 2.6) that $_Q^n\Pi^s_{D,\mathbf{D},V,\mathbf{v}}$ is a locally closed subset of $_Q\Lambda^s_{D,V}$. Let

$$_{QQ'}^n\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{v}} = {}_Q^n\Pi^s_{D,\mathbf{D},V,\mathbf{v}} \cap {}_{QQ'}\Lambda^{s,*s}_{D,V} \ .$$

Then $_{QQ'}^n\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{v}}$ is an open subset of $_Q^n\Pi^s_{D,\mathbf{D},V,\mathbf{v}}$ and therefore it is empty or has pure dimension equal to that of $_Q^n\Pi^s_{D,\mathbf{D},V,\mathbf{v}}$. Let $_{QQ'}^n\mathcal{T}(d,\mathbf{d},v,\mathbf{v})$ be the set of irreducible components of $_{QQ'}^n\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{v}}$.

The restriction of the map $\gamma: {}_Q\Lambda_{D,V,U} \to {}_Q\Lambda_U \times {}_Q\Lambda^{*s}_{D,T}$ (cf. 2.5) to $_{QQ'}\Lambda^{s,*s}_{D,V,U} = {}_{QQ'}\Lambda^{s,*s}_{D,V} \cap {}_Q\Lambda_{D,V,U}$ has the image equal to $_Q^{\delta}\Lambda'_U \times {}_Q\Lambda^{s,*s}_{D,T}$, where δ is as in 2.5.b, and $_Q^{\delta}\Lambda'_U$ denotes the open subset of $_Q^{\delta}\Lambda_U$ consisting of all $x \in {}_Q^{\delta}\Lambda_U$ such that $\cap_{\substack{h \in H \\ \text{Out}(h)=i}} \ker x_h = \{0\}$ for any $i \in I'$. It follows that the restriction of the map γ to $_{QQ'}^n\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{v}} \cap {}_Q\Lambda_{D,V,U}$ has the image equal to $_Q^{\delta}\Lambda'_U \times {}_Q^n\Pi^{s,*s}_{D,\mathbf{D},T,\mathbf{v}}$, and one can repeat the argument in 2.13 to get a bijection

$$_{QQ'}\beta_n: {}_{QQ'}^n\mathcal{T}(d,\mathbf{d},v,\mathbf{v}) \xrightarrow{\sim} \bigsqcup_{v_0 \in \mathbb{Z}_{\geq 0}[I]} {}_Q^n\mathcal{S}(d,\mathbf{d},v_0,\mathbf{v}) \times {}_{QQ'}\mathcal{M}(d,v_0,v) \ .$$

3.5. **Levi restriction and the first bijection for tensor product varieties.** (cf. 2.10)

Let $\mathbf{v}$, $\tilde{\mathbf{v}}$ be n-tuples of elements of $\mathbb{Z}[I]$, $\hat{\mathbf{v}}$ be an n-tuple of elements of $\mathbb{Z}[I'] \subset \mathbb{Z}[I]$, such that $\sum_{k=1}^n \mathbf{v}^k + \sum_{k=1}^n \tilde{\mathbf{v}}^k + \sum_{k=1}^n \hat{\mathbf{v}}^k = \dim V$, and let $_{QQ'}^n\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{v},\tilde{\mathbf{v}},\hat{\mathbf{v}}}$ be a locally closed subset of $_{QQ'}^n\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{v}} \cap {}_Q^n\Pi^s_{D,\mathbf{D},V,\mathbf{v},\tilde{\mathbf{v}}+\hat{\mathbf{v}}}$, consisting of all $(x,p,q) \in {}_{QQ'}^n\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{v}} \cap {}_Q^n\Pi^s_{D,\mathbf{D},V,\mathbf{v},\tilde{\mathbf{v}}+\hat{\mathbf{v}}}$ such that the dimension of the maximal graded subspace $\hat{\mathbf{V}}^k$ of $\overline{p(\mathbf{D}^k)}$ satisfying the following conditions (cf. 3.3.a)

$$
\begin{aligned}
&\hat{\mathbf{V}}_i^k \subset \left(\overline{p(\mathbf{D}^{k-1})}\right)_i & &\text{for any } i \notin I' \ , \\
\text{(3.5.a)} \quad &x_h \hat{\mathbf{V}}^k \subset \hat{\mathbf{V}}^k & &\text{for any } h \in H' \ , \\
&x_h \hat{\mathbf{V}}^k \subset \overline{p(\mathbf{D}^{k-1})} & &\text{for any } h \in H^{QQ'} \ , \\
&q_i(\hat{\mathbf{V}}_i^k) \subset \mathbf{D}_i^{k-1} & &\text{for any } i \in I' \ ,
\end{aligned}
$$

is equal $\sum_{l=1}^{k-1} \mathbf{v}^l + \sum_{l=1}^{k-1} \tilde{\mathbf{v}}^l + \sum_{l=1}^k \hat{\mathbf{v}}^l$ (cf. the definition of $_Q^n\Pi^s_{D,\mathbf{D},V,\mathbf{v},\tilde{\mathbf{v}}}$ in 2.6). Note that the maximal subspace $\hat{\mathbf{V}}^k$ satisfying conditions 3.5.a contains $\overline{p(\mathbf{D}^{k-1})}$, which has dimension $\sum_{l=1}^{k-1} \mathbf{v}^l + \sum_{l=1}^{k-1} \tilde{\mathbf{v}}^l + \sum_{l=1}^{k-1} \hat{\mathbf{v}}^l$, and is contained in $\overline{q^{-1}(\mathbf{D}^{k-1})}$, which has dimension $\sum_{l=1}^{k-1} \mathbf{v}^l + \sum_{l=1}^k \tilde{\mathbf{v}}^l + \sum_{l=1}^k \hat{\mathbf{v}}^l$.

Let $\mathbf{V} = (0 = \mathbf{V}^0 = \hat{\mathbf{V}}^1 \subset \tilde{\mathbf{V}}^1 \subset \mathbf{V}^1 \subset \hat{\mathbf{V}}^2 \subset \tilde{\mathbf{V}}^2 \subset \ldots \subset \hat{\mathbf{V}}^n \subset \tilde{\mathbf{V}}^n \subset \mathbf{V}^n = V)$ be a $3n$-step $\mathbb{Z}[I]$-graded partial flag in V, such that $\dim \hat{\mathbf{V}}^k - \dim \mathbf{V}^{k-1} = \hat{\mathbf{v}}^k$, $\dim \tilde{\mathbf{V}}^k - \dim \hat{\mathbf{V}}^k = \tilde{\mathbf{v}}^k$, $\dim \mathbf{V}^k - \dim \tilde{\mathbf{V}}^k = \mathbf{v}^k$, and let $_{QQ'}^n\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{V}}$ be a locally closed subset of $_{QQ'}^n\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{v}}$ consisting of all $(x,p,q) \in {}_{QQ'}^n\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{v}}$ such that $\mathbf{V}^k = \overline{p(\mathbf{D}^k)}$, $\tilde{\mathbf{V}}^k = \overline{q^{-1}(\mathbf{D}^{k-1})} \cap \mathbf{V}^k$, and $\hat{\mathbf{V}}^k$ is the maximal graded subspace of $\mathbf{V}^k$ satisfying conditions 3.5.a (cf. the definition of $_Q^n\Pi^s_{D,\mathbf{D},V,\mathbf{V}}$ in 2.6).

The variety ${}^n_{QQ'}\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{v},\tilde{\mathbf{v}},\hat{\mathbf{v}}}$ is a fibration over the variety of $3n$-step partial flags in V with dimensions of the subfactors given by $\mathbf{v}$, $\tilde{\mathbf{v}}$, and $\hat{\mathbf{v}}$, and a fiber of this fibration is isomorphic to ${}^n_{QQ'}\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{V}}$.

One has a regular map (given by restrictions of x, p, and q)

$$
{}_{QQ'}\sigma_n: \quad {}^n_{QQ'}\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{V}} \to
$$
$$
\to {}_{QQ'}\Lambda^{s,*s}_{\mathbf{D}^1/\mathbf{D}^0,\mathbf{V}^1/\hat{\mathbf{V}}^1,\tilde{\mathbf{V}}^1/\hat{\mathbf{V}}^1} \times \ldots \times {}_{QQ'}\Lambda^{s,*s}_{\mathbf{D}^n/\mathbf{D}^{n-1},\mathbf{V}^n/\hat{\mathbf{V}}^n,\tilde{\mathbf{V}}^n/\hat{\mathbf{V}}^n} \times
$$
$$
\times {}_{Q'}\Lambda_{\hat{\mathbf{V}}^1/\mathbf{V}^0} \times \ldots \times {}_{Q'}\Lambda_{\hat{\mathbf{V}}^n/\mathbf{V}^{n-1}} \, .
$$

Let D', V' be $\mathbb{Z}[I']$-graded $\mathbb{C}$-linear spaces defined in 3.2, $\mathbf{D}' = (0 = \mathbf{D}'^0 \subset \mathbf{D}'^1 \subset \ldots \subset \mathbf{D}'^n = D')$ (resp. $\mathbf{V}' = (0 = \mathbf{V}'^0 = \tilde{\mathbf{V}}'^1 \subset \mathbf{V}'^1 \subset \tilde{\mathbf{V}}'^2 \subset \ldots \subset \tilde{\mathbf{V}}'^n \subset \mathbf{V}'^n = V')$ be an n-step partial flag in D' (resp. a $2n$-step partial flag in V') induced by the flags $(0 = \mathbf{D}^0 \subset \mathbf{D}^1 \subset \ldots \subset \mathbf{D}^n = D)$ and $(0 = \mathbf{V}^0 \subset \mathbf{V}^1 \subset \ldots \subset \mathbf{V}^n = V)$ (resp. by the flag $(0 = \mathbf{V}^0 = \hat{\mathbf{V}}^1 \subset \mathbf{V}^1 \subset \hat{\mathbf{V}}^2 \subset \ldots \subset \hat{\mathbf{V}}^n \subset \mathbf{V}^n = V)$).

Recall (cf. 2.12) that

$$
{}_{Q'}\sigma_n: \quad {}^n_{Q'}\Pi^{s,*s}_{D',\mathbf{D}',V',\mathbf{V}'} \to
$$
$$
\to {}_{Q'}\Lambda^{s,*s}_{\mathbf{D}'^1/\mathbf{D}'^0,\mathbf{V}'^1/\tilde{\mathbf{V}}'^1} \times \ldots \times {}_{Q'}\Lambda^{s,*s}_{\mathbf{D}'^n/\mathbf{D}'^{n-1},\mathbf{V}'^n/\tilde{\mathbf{V}}'^n} \times
$$
$$
\times {}_{Q'}\Lambda_{\tilde{\mathbf{V}}'^1/\mathbf{V}'^0} \times \ldots \times {}_{Q'}\Lambda_{\tilde{\mathbf{V}}'^n/\mathbf{V}'^{n-1}}
$$

is the map obtained by restricting x, p, and q. Let $\zeta'_{QQ'}$ denote the restriction of the map $\zeta_{QQ'}$ (cf. 3.2) to ${}^n_{QQ'}\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{V}} \subset {}_Q\Lambda_{D,V}$.

The following proposition can be proven by an inductive (in n) argument similar to the ones used in subsections 2.7– 2.12. Since the proof is completely analogous to the proofs in these subsections it is omitted.

Proposition.   3.5.b. *The image of the map ${}_{QQ'}\sigma_n$ is equal to*

$$
{}_{QQ'}\Lambda^{s,*s}_{\mathbf{D}^1/\mathbf{D}^0,\mathbf{V}^1/\hat{\mathbf{V}}^1,\tilde{\mathbf{V}}^1/\hat{\mathbf{V}}^1} \times \ldots \times {}_{QQ'}\Lambda^{s,*s}_{\mathbf{D}^n/\mathbf{D}^{n-1},\mathbf{V}^n/\hat{\mathbf{V}}^n,\tilde{\mathbf{V}}^n/\hat{\mathbf{V}}^n} \times X \,,
$$

where X is the open subset of ${}_{Q'}\Lambda_{\hat{\mathbf{V}}^1/\mathbf{V}^0} \times \ldots \times {}_{Q'}\Lambda_{\hat{\mathbf{V}}^n/\mathbf{V}^{n-1}}$ such that

$$
{}_{Q'}\Lambda^{s,*s}_{\mathbf{D}'^1/\mathbf{D}'^0,\mathbf{V}'^1/\tilde{\mathbf{V}}'^1} \times \ldots \times {}_{Q'}\Lambda^{s,*s}_{\mathbf{D}'^n/\mathbf{D}'^{n-1},\mathbf{V}'^n/\tilde{\mathbf{V}}'^n} \times X
$$

is the image of the map ${}_{Q'}\sigma_n \circ \zeta'_{QQ'}$.

3.5.c. *The set ${}^n_{QQ'}\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{V}}$ is an open dense subset of the total space of a locally trivial fibration over the image of ${}_{QQ'}\sigma_n$ with a smooth connected fiber and such that the restriction of the projection map onto ${}^n_{QQ'}\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{V}}$ is equal to ${}_{QQ'}\sigma_n$.*

3.5.d. *The dimension of ${}^n_{QQ'}\Pi^{s,*s}_{D,\mathbf{D},V,\mathbf{v},\tilde{\mathbf{v}},\hat{\mathbf{v}}}$ does not depend on $\tilde{\mathbf{v}}$ and $\hat{\mathbf{v}}$ (it only depends on d, $\mathbf{d}$, v, and $\mathbf{v}$.*

The above proposition together with 2.12 implies that the map ${}_{QQ'}\sigma_n$ induces the following bijection of sets of irreducible components

(3.5.e) ${}_{QQ'}\alpha_n: \quad {}^n_{QQ'}\mathcal{T}(d,\mathbf{d},v,\mathbf{v}) \xrightarrow{\sim}$
$$
\xrightarrow{\sim} \bigsqcup_{\substack{\mathbf{u}^1,\ldots,\mathbf{u}^n \in \mathbb{Z}_{\geq 0}[I] \\ v-\mathbf{u}^1-\ldots-\mathbf{u}^n \in \mathbb{Z}_{\geq 0}[I']}} {}^n_{Q'}\mathcal{S}(\delta_{QQ'}(d,v),\delta_{QQ'}(\mathbf{d},\mathbf{u}),\rho_{QQ'}(v),\rho_{QQ'}(\mathbf{u})) \times
$$
$$
\times {}_{QQ'}\mathcal{M}(\mathbf{d}^1,\mathbf{v}^1,\mathbf{u}^1) \times \ldots \times {}_{QQ'}\mathcal{M}(\mathbf{d}^n,\mathbf{v}^n,\mathbf{u}^n) \,,
$$

where $\delta_{QQ'}(d,v)$, and $\rho_{QQ'}(v)$ are as in 3.3, and $\left(\delta_{QQ'}(\mathbf{d},\mathbf{v})\right)^k = \delta_{QQ'}(\mathbf{d}^k,\mathbf{v}^k)$, $\left(\rho_{QQ'}(\mathbf{v})\right)^k = \rho_{QQ'}(\mathbf{v}^k)$.

3.6. Levi restriction and the tensor product decomposition. Let ${}_{QQ'}\tau_n$ be the bijection

$$\bigsqcup_{\substack{\mathbf{u}^1,\dots,\mathbf{u}^n\in\mathbb{Z}_{\geq 0}[I] \\ v-\mathbf{u}^1-\dots-\mathbf{u}^n\in\mathbb{Z}_{\geq 0}[I']}} {}^n_{Q'}\mathcal{S}(\delta_{QQ'}(d,v),\delta_{QQ'}(\mathbf{d},\mathbf{u}),\rho_{QQ'}(v),\rho_{QQ'}(\mathbf{u})) \times$$

$$\times\, {}_{QQ'}\mathcal{M}(\mathbf{d}^1,\mathbf{v}^1,\mathbf{u}^1)\times\dots\times {}_{QQ'}\mathcal{M}(\mathbf{d}^n,\mathbf{v}^n,\mathbf{u}^n) \xrightarrow{\sim}$$

$$\xrightarrow{\sim} \bigsqcup_{v_0\in\mathbb{Z}_{\geq 0}[I]} {}^n_Q\mathcal{S}(d,\mathbf{d},v_0,\mathbf{v}) \times {}_{QQ'}\mathcal{M}(d,v_0,v)\,.$$

given by

$$ {}_{QQ'}\tau_n = {}_{QQ'}\beta_n \circ {}_{QQ'}\alpha_n^{-1}\,.$$

It follows from the definitions of the bijections ${}_{QQ'}\alpha_n$ (cf. 3.5), ${}_{QQ'}\beta_n$ (cf. 3.4), and $\theta_{QQ'}$ (cf. 3.3), that the following diagram of bijections is commutative:

(3.6.a)

$$\bigsqcup_{\substack{\mathbf{u}^1,\dots,\mathbf{u}^n\in\mathbb{Z}_{\geq0}[I]\\ \mathbf{u}^1+\dots+\mathbf{u}^n=v}} \qquad {}_Q\mathcal{M}(\mathbf{d}^1,\mathbf{v}^1,\mathbf{u}^1)\times\dots\times {}_Q\mathcal{M}(\mathbf{d}^n,\mathbf{v}^n,\mathbf{u}^n) \xrightarrow{\ {}_Q\tau_n\ }$$

$$\downarrow \theta_{QQ'}\times\dots\times\theta_{QQ'}$$

$$\bigsqcup_{\substack{\mathbf{u}^1,\dots,\mathbf{u}^n\in\mathbb{Z}_{\geq0}[I]\\ \mathbf{u}^1+\dots+\mathbf{u}^n=v\\ \mathbf{w}^1,\dots,\mathbf{w}^n\in\mathbb{Z}_{\geq0}[I']}} \quad \begin{aligned} &{}_{QQ'}\mathcal{M}(\mathbf{d}^1,\mathbf{v}^1,\mathbf{u}^1-\mathbf{w}^1)\times\\ &{}_{Q'}\mathcal{M}(\delta_{QQ'}(\mathbf{d}^1,\mathbf{u}^1),\rho_{QQ'}(\mathbf{u}^1-\mathbf{w}^1),\rho_{QQ'}(\mathbf{u}^1))\times\\ &\qquad\qquad\qquad\vdots\\ &\times{}_{QQ'}\mathcal{M}(\mathbf{d}^n,\mathbf{v}^n,\mathbf{u}^n-\mathbf{w}^n)\times\\ &{}_{Q'}\mathcal{M}(\delta_{QQ'}(\mathbf{d}^n,\mathbf{u}^n),\rho_{QQ'}(\mathbf{u}^n-\mathbf{w}^n),\rho_{QQ'}(\mathbf{u}^n)) \end{aligned}$$

$$\downarrow (\mathrm{Id}\times\dots\times\mathrm{Id}\times{}_{Q'}\tau_n)\circ P$$

$$\bigsqcup_{\substack{\mathbf{x}^1,\dots,\mathbf{x}^n\in\mathbb{Z}_{\geq0}[I]\\ v-\mathbf{x}^1-\dots-\mathbf{x}^n\in\mathbb{Z}_{\geq0}[I']\\ w\in\mathbb{Z}_{\geq0}[I']}} \quad \begin{aligned} &{}_{QQ'}\mathcal{M}(\mathbf{d}^1,\mathbf{v}^1,\mathbf{x}^1)\times\dots\times{}_{QQ'}\mathcal{M}(\mathbf{d}^n,\mathbf{v}^n,\mathbf{x}^n)\times\\ &{}_{Q'}^n\mathcal{S}(\delta_{QQ'}(d,v),\delta_{QQ'}(\mathbf{d},\mathbf{x}),\rho_{QQ'}(v-w),\rho_{QQ'}(\mathbf{x}))\times\\ &{}_{Q'}\mathcal{M}(\delta_{QQ'}(d,v),\rho_{QQ'}(v-w),\rho_{QQ'}(v)) \end{aligned}$$

$$\downarrow {}_{QQ'}\tau_n\times\mathrm{Id}$$

$$\bigsqcup_{\substack{v_0\in\mathbb{Z}_{\geq0}[I]\\ w\in\mathbb{Z}_{\geq0}[I']}} \quad \begin{aligned} &{}_Q^n\mathcal{S}(d,\mathbf{d},v_0,v)\times{}_{QQ'}\mathcal{M}(d,v_0,v-w)\times\\ &{}_{Q'}\mathcal{M}(\delta_{QQ'}(d,v),\rho_{QQ'}(v-w),\rho_{QQ'}(v)) \end{aligned}$$

$$\downarrow \mathrm{Id}\times\theta_{QQ'}^{-1}$$

$$\bigsqcup_{v_0\in\mathbb{Z}_{\geq0}[I]} \quad {}_Q^n\mathcal{S}(d,\mathbf{d},v_0,v)\times{}_Q\mathcal{M}(d,v_0,v)\longleftarrow$$

where P is the permutation that moves all even terms in the direct product to the right. Roughly speaking the commutativity of the above diagram means that "the Levi restriction commutes with the tensor product decomposition".

3.7. Digression: $\mathfrak{sl}_2$ case. The Levi restriction bijections allow one to reduce generic ADE case to the case of a quiver R with one vertex and no edges (which corresponds to $\mathfrak{sl}_2$). This subsection contains a complete description of quiver varieties and tensor product varieties for the quiver R.

Since R has only one vertex and no edges the vertex and edge indices are omitted in the notation. Thus given two (non graded) $\mathbb{C}$-linear spaces D and V with dimensions d and v respectively, ${}_R\Lambda_{D,V}$ is the variety of pairs (p,q), where $p\in\mathrm{Hom}_{\mathbb{C}}(D,V)$, $q\in\mathrm{Hom}_{\mathbb{C}}(V,D)$, and $pq=0$. The open set ${}_R\Lambda_{D,V}^s$ (resp. ${}_R\Lambda_{D,V}^{*s}$) consists of all $(p,q)\in{}_R\Lambda_{D,V}$ such that p is surjective (resp. q is injective).

The map $(p, q) \rightarrow qp \in \mathrm{End}_{\mathbb{C}}(D)$ provides an isomorphism between the quiver variety ${}_R\mathfrak{M}_{D,V}^{s,*s} = {}_R\Lambda_{D,V}^{s,*s}/G_V$ (cf. 2.3) and the GL_D-orbit in $\mathrm{End}_{\mathbb{C}}(D)$ consisting of all $t \in \mathrm{End}_{\mathbb{C}}(D)$ such that $t^2 = 0$ and $\mathrm{rank}\, t = v$. Note that this orbit is empty (if $2v > d$) or is a smooth connected quasi-projective variety of dimension $2v(d-v)$ (if $2v \leq d$), which confers the corresponding statements in 2.3.

Similarly, the map $(p, q) \rightarrow (qp, \ker p)$ provides an isomorphism between the quiver variety ${}_R\mathfrak{M}^s(d, v_0, v) = {}_R\Lambda_{D,V,v_0}^s/G_V$ (cf. 2.4) and the variety of pairs (t, B), where $t \in \mathrm{End}_{\mathbb{C}}(D)$, $t^2 = 0$, $\mathrm{rank}\, t = v_0$, and B is a subspace of D such that $\mathrm{im}\, t \subset B \subset \ker t$, $\dim B = d - v$. It follows that ${}_R\mathfrak{M}^s(d, v_0, v)$ is smooth connected if $v_0 \leq v \leq d-v_0$ and empty otherwise, and hence its set of irreducible components ${}_R\mathcal{M}(d, v_0, v)$ is a one-element or the empty set respectively. Therefore

$$ {}_R\mathcal{M}(d, v_0) = \bigsqcup_{v \in \mathbb{Z}_{\geq 0}} {}_R\mathcal{M}(d, v_0, v) = \bigsqcup_{v=v_0}^{d-v_0} {}_R\mathcal{M}(d, v_0, v) = \bigsqcup_{v=v_0}^{d-v_0} \{{}_R\mathfrak{M}^s(d, v_0, v)\} \ . $$

Endow this set with the following structure of an sl_2-crystal:

$$ {}_R wt({}_R\mathfrak{M}^s(d, v_0, v)) = (d - 2v) \ , $$

$$ {}_R\varepsilon({}_R\mathfrak{M}^s(d, v_0, v)) = v - v_0 \ , $$

$$ {}_R\varphi({}_R\mathfrak{M}^s(d, v_0, v)) = d - v - v_0 \ , $$

(3.7.a)
$$ {}_R\tilde{e}({}_R\mathfrak{M}^s(d, v_0, v)) = \begin{cases} {}_R\mathfrak{M}^s(d, v_0, v-1) & \text{if } v > v_0 \ , \\ 0 & \text{if } v \leq v_0 \ , \end{cases} $$

$$ {}_R\tilde{f}({}_R\mathfrak{M}^s(d, v_0, v)) = \begin{cases} {}_R\mathfrak{M}^s(d, v_0, v+1) & \text{if } v < d - v_0 \ , \\ 0 & \text{if } v \geq d - v_0 \ . \end{cases} $$

Here the weight lattice of $\mathfrak{sl}_2$ is identified with $\mathbb{Z}$, and the indexes of ε, φ, $\tilde{e}$, and $\tilde{f}$ are omitted because $\mathfrak{sl}_2$ has only one root. The set ${}_R\mathcal{M}(d, v_0)$ equipped with the above structure is a highest weight normal sl_2-crystal with the highest weight $(d - 2v_0)$. In other words, it is isomorphic (as a crystal) to $\mathcal{L}((d - 2v_0))$ (the crystal of the canonical basis of the highest weight irreducible representation with highest weight $(d - 2v_0)$).

Let D^1 be a subspace of D, $\mathbf{D}$ be the 2-step partial flag $(0 \subset D^1 \subset D)$, $\mathbf{d}$ be the pair $(\mathbf{d}^1, \mathbf{d}^2)$, where $\mathbf{d}^1 = \dim D^1$, $\mathbf{d}^2 = d - \dim D^1$, and $\mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2)$ be an element of $\mathbb{Z} \oplus \mathbb{Z}$. The map $(p, q) \rightarrow qp \in \mathrm{End}_{\mathbb{C}}(D)$ provides an isomorphism between the multiplicity variety ${}_R^2\mathfrak{S}_{D,\mathbf{D},V,\mathbf{v}}$ (cf. 2.6) and the variety of all $t \in \mathrm{End}_{\mathbb{C}}(D)$ such that

$$ t^2 = 0 \ , $$

$$ tD^1 \subset D^1 \ , $$

(3.7.b)
$$ \mathrm{rank}\, t = v \ , $$

$$ \mathrm{rank}\, t|_{D^1} = \mathbf{v}^1 \ , $$

$$ \mathrm{rank}\, t|_{(D/D^1)} = \mathbf{v}^2 \ . $$

Similarly, the map $(p, q) \rightarrow (qp, \ker p)$ provides an isomorphism between the tensor product variety ${}_R^2\mathfrak{T}_{D,\mathbf{D},V,\mathbf{v}}$ (cf. 2.6) and the variety of pairs (t, B), where $t \in \mathrm{End}_{\mathbb{C}}(D)$ satisfies conditions (3.7.b) and B is a subspace of D such that $\mathrm{im}\, t \subset B \subset \ker t$, $\dim B = d - v$.

A straightforward linear algebra considerations (cf. [Mal00]) show that the variety of all $t \in \mathrm{End}_{\mathbb{C}}(D)$ satisfying conditions (3.7.b) is empty if either $v < \mathbf{v}^1 + \mathbf{v}^2$, or $v > \mathbf{d}^2 - \mathbf{v}^2 + \mathbf{v}^1$, or $v > \mathbf{d}^1 - \mathbf{v}^1 + \mathbf{v}^2$, and is a smooth connected quasi-projective variety otherwise. Hence the RHS of the tensor decomposition bijection $_R\tau_2$ (cf. 2.14) becomes

$$\bigsqcup_{v_0 \in \mathbb{Z}_{\geq 0}} {}^2_R\mathcal{S}(d, \mathbf{d}, v_0, \mathbf{v}) \times {}_R\mathcal{M}(d, v_0) = \bigsqcup_{v_0 = \mathbf{v}^1 + \mathbf{v}^2}^{\min(\mathbf{d}^2 - \mathbf{v}^2 + \mathbf{v}^1, \mathbf{d}^1 - \mathbf{v}^1 + \mathbf{v}^2)} {}_R\mathcal{M}(d, v_0) \ .$$

The bijection

$$_R\tau_2: \ {}_R\mathcal{M}(\mathbf{d}^1, \mathbf{v}^1) \times {}_R\mathcal{M}(\mathbf{d}^2, \mathbf{v}^2) \xrightarrow{\sim} \bigsqcup_{v_0 = \mathbf{v}^1 + \mathbf{v}^2}^{\min(\mathbf{d}^2 - \mathbf{v}^2 + \mathbf{v}^1, \mathbf{d}^1 - \mathbf{v}^1 + \mathbf{v}^2)} {}_R\mathcal{M}(d, v_0)$$

can be written as

(3.7.c) $\qquad _R\tau_2\big({}_R\mathfrak{M}(\mathbf{d}^1, \mathbf{v}^1, \mathbf{u}^1), {}_R\mathfrak{M}(\mathbf{d}^2, \mathbf{v}^2, \mathbf{u}^2)\big) = {}_R\mathfrak{M}(d, v_0, \mathbf{u}^1 + \mathbf{u}^2) \ ,$

where v_0 is described as follows. Fix a subspace B in D such that $\dim B = d - \mathbf{u}^1 - \mathbf{u}^2$ and $\dim D^1 \cap B = \mathbf{d}^1 - \mathbf{u}^1$. Then v_0 is the (unique) integer such that the set of operators $t \in \mathrm{End}_{\mathbb{C}}(D)$ with $\mathrm{rank}\, t = v_0$ form an open subset in the set of all t satisfying the following conditions:

(3.7.d)
$$\begin{aligned} &t^2 = 0 \ , \\ &tD^1 \subset D^1 \ , \\ &\mathrm{rank}\, t|_{D^1} = \mathbf{v}^1 \ , \\ &\mathrm{rank}\, t|_{(D/D^1)} = \mathbf{v}^2 \ , \\ &\mathrm{im}\, t \subset B \subset \ker t \ . \end{aligned}$$

Again elementary linear algebra (cf. [Mal00]) shows that

(3.7.e) $\qquad\qquad\qquad v_0 = \min(\mathbf{u}^2 + \mathbf{v}^1, \mathbf{d}^1 - \mathbf{u}^1 + \mathbf{v}^2) \ .$

The equality (3.7.e) together with definition of the tensor product of crystals (cf. 1.3) imply the following theorem (cf. [Mal00]).

Theorem. *The bijection $_R\tau_2$ is a crystal isomorphism*

$$_R\tau_2: \quad {}_R\mathcal{M}(\mathbf{d}^1, \mathbf{v}^1) \otimes {}_R\mathcal{M}(\mathbf{d}^2, \mathbf{v}^2) \xrightarrow{\sim} \bigoplus_{v_0 \in \mathbb{Z}_{\geq 0}} {}^2_R\mathcal{S}(d, \mathbf{d}, v_0, \mathbf{v}) \otimes {}_R\mathcal{M}(d, v_0) \ ,$$

where $\mathbf{d} = (\mathbf{d}^1, \mathbf{d}^2)$, $\mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2)$, $d = \mathbf{d}^1 + \mathbf{d}^2$, and the set ${}^2_R\mathcal{S}(d, \mathbf{d}, v_0, \mathbf{v})$ is considered as a trivial $\mathfrak{sl}_2$-crystal.

Remark. In [Mal00] definitions of tensor product and multiplicity varieties for $\mathfrak{sl}_2$ are slightly different. Namely the varieties in [Mal00] are fibrations over the Grassmannian of all subspaces D^1 in D with given dimension, and the fibers of these fibrations are the corresponding (tensor product or multiplicity) varieties as they defined in this paper. The sets of irreducible components and the tensor decomposition bijection are the same here and in [Mal00].

3.8. **A crystal structure on $_Q\mathcal{M}(d, v_0)$.** In this subsection it is shown that the set $_Q\mathcal{M}(d, v_0)$ (cf. 2.5) can be endowed with a structure of $\mathfrak{g}$-crystal. This structure was introduced by Nakajima [Nak98] following an idea of Lusztig [Lus91, Section 12].

The first step is to define the weight function. Identify the weight lattice of $\mathfrak{g}$ with $\mathbb{Z}[I]$ (i.e. $i \in I \subset \mathbb{Z}[I]$ is the i-th fundamental weight). Then the weight function

$$_Qwt: \ _Q\mathcal{M}(d, v_0) \to \mathbb{Z}[I]$$

is given by

$$_Qwt(Z) = d - 2v + _QXv$$

for $Z \in \ _Q\mathcal{M}(d, v_0, v) \subset \ _Q\mathcal{M}(d, v_0)$. Here $_QX$ is the matrix defined in (2.1.a). The crucial property of the function $_Qwt$ is its good behavior with respect to the Levi restriction. Namely

$$\rho_{QQ'}(d - 2v + _QXv) = \delta_{QQ'}(d, v) - 2\rho_{QQ'}(v) + _{Q'}X\rho_{QQ'}(v) \ ,$$

where $\delta_{QQ'}$ and $\rho_{QQ'}$ are as in 3.3. This allows one to use Levi restriction to $\mathfrak{sl}_2$ subalgebras to define the crystal structure on $_Q\mathcal{M}(d, v_0)$. More explicitly, let R_i be the subquiver of Q consisting of the vertex i and no edges. Recall (cf. 3.3) that there is a bijection

$$(3.8.a) \qquad \theta_{QR_i}: \ _Q\mathcal{M}(d, v_0) \xrightarrow{\sim} \bigsqcup_{v \in \mathbb{Z}_{\geq 0}[I]} {}_{QR_i}\mathcal{M}(d, v_0, v) \times {}_{R_i}\mathcal{M}(d_i + (_QXv)_i, v_i) \ .$$

Consider the RHS of (3.8.a) as an $\mathfrak{sl}_2$-crystal with the crystal structure coming from the second multiple (on which it is defined as in 3.7), and let

$$(3.8.b) \qquad \begin{aligned} _Q\varepsilon_i &= {}_{R_i}\varepsilon \circ \theta_{QR_i} \ , \\ _Q\varphi_i &= {}_{R_i}\varphi \circ \theta_{QR_i} \ , \\ _Q\tilde{e}_i &= \theta_{QR_i}^{-1} \circ {}_{R_i}\tilde{e} \circ \theta_{QR_i} \ , \\ _Q\tilde{f}_i &= \theta_{QR_i}^{-1} \circ {}_{R_i}\tilde{f} \circ \theta_{QR_i} \ . \end{aligned}$$

These formulas together with the weight function $_Qwt$ provide a structure of $\mathfrak{g}$-crystal on the set $_Q\mathcal{M}(d, v_0)$. By abuse of notation this crystal is also denoted by $_Q\mathcal{M}(d, v_0)$.

Proposition. 3.8.c. *If $d - 2v_0 + _QXv_0 \notin \mathbb{Z}_{\geq 0}[I]$ then $_Q\mathcal{M}(d, v_0)$ is an empty set.*
 3.8.d. *If $d - 2v_0 + _QXv_0 \in \mathbb{Z}_{\geq 0}[I]$ then $_Q\mathcal{M}(d, v_0)$ is a highest weight normal $\mathfrak{g}$-crystal with the highest weight $d - 2v_0 + _QXv_0$.*
 3.8.e. *If $d - 2v_0 + _QXv_0 = d' - 2v_0' + _QXv_0'$ then the crystals $_Q\mathcal{M}(d, v_0)$ and $_Q\mathcal{M}(d', v_0')$ are isomorphic.*

Proof. 3.8.c follows from 2.3.d.

3.8.d follows directly from definitions and the fact that $_Q\mathfrak{M}^s(d, v_0, v_0)$ (a connected smooth variety) is the only element of $_Q\mathcal{M}(d, v_0)$ which is killed by $_Q\tilde{e}_i$ for all $i \in I$. To prove this let $Z \in \ _Q\mathcal{M}(d, v_0)$ be an irreducible component of $_Q\Lambda^s_{D, V, U}$, and (x, p, q) be a generic point of Z. Let γ be as in 2.5 and $(x^{UU}, (x^{TT}, p^{TD}, q^{DT})) = \gamma((x, p, q))$. If $U \neq \{0\}$ then there exists $i \in I$ such that $\bigcap_{\substack{h \in H \\ \mathrm{Out}(h)=i}} \ker x_h^{UU} \neq \{0\}$ (cf. [Lus91, 12]). It follows that $(\bigcap_{\substack{h \in H \\ \mathrm{Out}(h)=i}} \ker x_h) \bigcap \ker q_i \neq \{0\}$, and hence

$_Q\tilde{e}_i Z \neq 0$. Therefore if $_Q\tilde{e}_i Z = 0$ for all $i \in I$ then $U = \{0\}$, which means $Z = {}_Q\mathfrak{M}^s(d, v_0, v_0)$.

To prove 3.8.e note that as sets both $_Q\mathcal{M}(d, v_0, v_0 + u)$ and $_Q\mathcal{M}(d', v_0', v_0' + u)$ are in natural bijections (induced by the vector bundles γ – cf. 2.5) with the set of irreducible components of the variety $^\delta\Lambda_U$, where $\dim U = u$, and $\delta = d - 2v_0 + {}_QXv_0 = d' - 2v_0' + {}_QXv_0'$. In this way one obtains a bijection between $_Q\mathcal{M}(d, v_0, v_0 + u)$ and $_Q\mathcal{M}(d', v_0', v_0' + u)$, and it follows from definitions that this bijection is a crystal isomorphism. $\qquad\square$

3.9. **The main theorem.** From now on the quiver Q is fixed and thus omitted in the notation. The following is the main result of this paper.

Theorem. *The bijection τ_n (cf. 2.14) is a crystal isomorphism*

$$\tau_n: \quad \mathcal{M}(\mathbf{d}^1, \mathbf{v}^1) \otimes \ldots \otimes \mathcal{M}(\mathbf{d}^n, \mathbf{v}^n) \xrightarrow{\sim} \bigoplus_{v_0 \in \mathbb{Z}_{\geq 0}[I]} {}^n\mathcal{S}(d, \mathbf{d}, v_0, \mathbf{v}) \otimes \mathcal{M}(d, v_0) \, ,$$

where the set $^n\mathcal{S}(d, \mathbf{d}, v_0, \mathbf{v})$ is considered as a trivial $\mathfrak{g}$-crystal.

Note that both sides might be empty.

Proof. Because of commutativity of diagram (2.14.d) it is enough to consider the case $n = 2$, and because the definition of the crystal structure uses Levi restriction which commutes with tensor product (cf. (3.6.a)) it is enough to consider the case $\mathfrak{g} = \mathfrak{sl}_2$. Now the theorem follows from Theorem 3.7. $\qquad\square$

3.10. **Corollary.** Recall that $L(\mu)$ denotes the highest weight irreducible representation of $\mathfrak{g}$ with highest weight $\mu \in \mathbb{Z}_{\geq 0}[I]$, and $\mathcal{L}(\mu)$ denotes the crystal of the canonical basis of $L(\mu)$.

The following proposition is a corollary of Theorem 3.9.

Proposition.

3.10.a. *The $\mathfrak{g}$-crystal $\mathcal{M}(d, v_0)$ is isomorphic to $\mathcal{L}(\mu)$, where $\mu = d - 2v_0 + Xv_0$.*

3.10.b. *The cardinal of the set $^n\mathcal{S}(d, \mathbf{d}, v_0, \mathbf{v})$ of irreducible components of the multiplicity variety $^n\mathfrak{S}(d, \mathbf{d}, v_0, \mathbf{v})$ is equal to the multiplicity of $L(\mu)$ in $L(\mu^1) \otimes \ldots \otimes L(\mu^n)$, where $\mu = d - 2v_0 + Xv_0$, $\mu^k = \mathbf{d}^k - 2\mathbf{v}^k + X\mathbf{v}^k$. In other words,*

$$|^n\mathcal{S}(d, \mathbf{d}, v_0, \mathbf{v})| = \dim_\mathbb{C} \mathrm{Hom}_\mathfrak{g}(L(\mu), L(\mu^1) \otimes \ldots \otimes L(\mu^n)) \, .$$

3.10.c. *The bijection α_n (cf. 2.10) identifies the set $^n\mathcal{T}(d, \mathbf{d}, v, \mathbf{v})$ of irreducible components of the tensor product variety $^n\mathfrak{T}(d, \mathbf{d}, v, \mathbf{v})$ with the weight subset of weight $d - 2v + Xv$ in the crystal $\mathcal{L}(\mu^1) \otimes \ldots \otimes \mathcal{L}(\mu^n)$, where $\mu^1, \ldots, \mu^n$ are as in 3.10.b.*

Proof. The proposition follows from Theorem 3.9, Theorem 1.5, Proposition 3.8, and Proposition 2.12. $\qquad\square$

Remark. Note that 3.10.b is a generalization of a theorem due to Hall [Hal59] (cf. [Mac95, Chapter II]). Statement 3.10.a was also proven by Saito using results of [KS97].

3.11. **The extended Lie algebra $\mathfrak{g}'$.** Proposition 3.8.e shows that in the set of $\mathfrak{g}$-crystals $\{\mathcal{M}(d, v_0)\}_{d,v_0 \in \mathbb{Z}_{\geq 0}[I]}$ there are isomorphic elements, which suggests that in the context of quiver varieties it is more natural to consider a central extension of $\mathfrak{g}$ than $\mathfrak{g}$ itself.

Let $\mathfrak{g}' = \mathfrak{g} \oplus \mathfrak{t}$, where $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. Then $\mathfrak{g}'$ is a reductive Lie algebra. Identify the weight lattice of $\mathfrak{g}'$ with $\mathbb{Z}[I] \oplus \mathbb{Z}[I]$ in such a way that the projection $\mathbb{Z}[I] \oplus \mathbb{Z}[I] \to \mathbb{Z}[I]$ given by $(v, u) \to v - u$ is the projection from the weight lattice of $\mathfrak{g}'$ onto the weight lattice of $\mathfrak{g}$. A weight $(v, u) \in \mathbb{Z}[I] \oplus \mathbb{Z}[I]$ is called *integrable* if $u \in \mathbb{Z}_{\geq 0}[I]$ and $v - u \in \mathbb{Z}_{\geq 0}[I]$. Let $\mathcal{Q}^+ \subset \mathbb{Z}_{\geq 0}[I] \oplus \mathbb{Z}_{\geq 0}[I]$ be the set of all integrable weights. A finite dimensional representation of $\mathfrak{g}'$ is called *integrable* if the highest weights of all its irreducible components are integrable. Note that the category of finite dimensional integrable representations is closed with respect to tensor products.

One can endow the set $\mathcal{M}(d, v_0)$ with a structure of $\mathfrak{g}'$-crystal as follows. The maps ε_i, φ_i, $\tilde{e}_i$, and $\tilde{f}_i$ are given by (3.8.b) (note that roots of $\mathfrak{g}'$ are roots of $\mathfrak{g}$). The weight function is given by (cf. 3.8)

$$(3.11.a) \qquad wt'(Z) = (d - v + Xv, v) \in \mathbb{Z}[I] \oplus \mathbb{Z}[I]$$

for $Z \in \mathcal{M}(d, v_0, v) \subset \mathcal{M}(d, v_0)$. Note that this weight function commutes with the Levi restriction (cf. an analogous statement for the weight function wt in 3.8).

The following theorem shows that non-empty elements of the set

$$\{\mathcal{M}(d, v_0)\}_{(d,v_0) \in \mathbb{Z}_{\geq 0}[I] \oplus \mathbb{Z}_{\geq 0}[I]}$$

form a closed (with respect to tensor product) family of $\mathfrak{g}'$-crystals, which is isomorphic to the family of crystals of canonical bases of irreducible integrable representations of $\mathfrak{g}'$.

Theorem. 3.11.b. *If $(d - v_0 + Xv_0, v_0) \notin \mathcal{Q}^+$ then the set $\mathcal{M}(d, v_0)$ is empty.*
   3.11.c. *If $(d - v_0 + Xv_0, v_0) \in \mathcal{Q}^+$ then the set $\mathcal{M}(d, v_0)$ together with the weight function wt' and the maps ε_i, φ_i, $\tilde{e}_i$, and $\tilde{f}_i$ given by (3.8.b) is a highest weight normal $\mathfrak{g}'$-crystal with the highest weight $(d - v_0 + Xv_0, v_0)$.*
   3.11.d. *The image of the weight function $wt' : \mathcal{M}(d, v_0) \to \mathbb{Z}[I] \oplus \mathbb{Z}[I]$ is contained in $\mathbb{Z}_{\geq 0}[I] \times \mathbb{Z}_{\geq 0}[I]$.*
   3.11.e. *The bijection τ_n (cf. 2.14) is an isomorphism of $\mathfrak{g}'$-crystals*

$$\tau_n : \mathcal{M}(\mathbf{d}^1, \mathbf{v}^1) \otimes \ldots \otimes \mathcal{M}(\mathbf{d}^n, \mathbf{v}^n) \xrightarrow{\sim} \bigoplus_{v_0 \in \mathbb{Z}_{\geq 0}[I]} {}^n\mathcal{S}(d, \mathbf{d}, v_0, \mathbf{v}) \otimes \mathcal{M}(d, v_0)$$

where the set ${}^n\mathcal{S}(d, \mathbf{d}, v_0, \mathbf{v})$ is considered as a trivial $\mathfrak{g}'$-crystal.
   3.11.f. *The family $\{\mathcal{M}(d, v_0)\}$ of $\mathfrak{g}'$-crystals is isomorphic to the family $\{\mathcal{L}(d, v_0)\}$ of crystals of canonical bases of irreducible integrable representations of $\mathfrak{g}'$.*

In 3.11.f $\mathcal{L}(d, v_0)$ denotes the crystal of the canonical basis of the irreducible integrable representation $L(d, v_0)$ with the highest weight $(d - v_0 + Xv_0, v_0)$.

Proof. Proofs of 3.11.b, 3.11.c, 3.11.e, and 3.11.f are analogous to the proofs of Propositions 3.8.c, 3.8.d, Theorem 3.9, and Corollary 3.10.a, respectively, or one can deduce the former statements from the latter.

To prove 3.11.d note that an element of the set $\mathcal{M}(d, v_0, v)$ is an irreducible component of the variety $\mathfrak{M}^s_{D,V,v_0}$ (cf. 2.4), where $\dim D = d$, $\dim V = v$. A point

of this variety is a stable triple (x, p, q). Since it is stable im $p_i + \sum_{\substack{h \in H \\ \text{In}(h)=i}}$ im $x_h = V_i$. Therefore $v_i \leq (d + Xv)_i$, which implies 3.11.d. □

Remark. It follows from 3.11.d that all weights of a finite dimensional integrable representation of $\mathfrak{g}'$ belong to $\mathbb{Z}_{\geq 0}[I] \times \mathbb{Z}_{\geq 0}[I]$.

APPENDIX A. ANOTHER DESCRIPTION OF MULTIPLICITY VARIETIES AND THE TENSOR PRODUCT DIAGRAM

A.1. **Lusztig's description of the variety $\mathfrak{M}_{D,V}^{s,*s}$.** Lusztig (cf. [Lus98, Lus00a, Lus00b]) identified varieties $\mathfrak{M}_{D,V}^{s,*s}$ (for a fixed D and varying V) with locally closed subsets in a certain variety Z_D. This subsection contains an overview of the Lusztig's construction (see the above references for more details).

Recall (cf. 3.7) that in the $\mathfrak{sl}_2$ case the map $(p, q) \to qp$ allows one to identify $\mathfrak{M}_{D,V}^{s,*s}$ with a nilpotent orbit in $\text{End}_{\mathbb{C}} D$ (namely with the orbit consisting of all operators t such that $t^2 = 0$, and rank $t = \dim V$). Lusztig's construction is the direct generalization of this device. However for a generic quiver one has to take care not only of the maps p and q, but also of x (i.e one has to include the path algebra $\mathcal{F}$ into the play).

Let $\mathbb{C}^I = \oplus_{i \in I} \mathbb{C}$ considered as a semi-simple $\mathbb{C}$-algebra. Note that a $\mathbb{Z}[I]$-graded $\mathbb{C}$-linear space is the same as a $\mathbb{C}^I$-module, and that the path algebra $\mathcal{F}$ is a $\mathbb{C}^I$–$\mathbb{C}^I$ algebra. Let $\tilde{\mathcal{F}}$ be a $\mathbb{C}$-algebra defined as follows. As a $\mathbb{C}$-linear space $\tilde{\mathcal{F}} = \mathcal{F} \oplus \oplus_{i \in I} \mathbb{C} u_i$, and the multiplication $\circ$ in $\tilde{\mathcal{F}}$ is given by

$$f \circ f' = \sum_{i \in I} f \cdot \theta_i \cdot f' \, ,$$
$$f \circ u_i = f \cdot [i] \, ,$$
$$u_i \circ f = [i] \cdot f \, ,$$
$$u_i \circ u_j = \delta_{ij} \, u_i \, ,$$

where $f, f' \in \mathcal{F}$, $i, j \in I$, θ_i is as in 2.1.b, $[i] \in \mathcal{F}$ denotes the path of length 0 starting and ending at a vertex $i \in I$, and $\cdot$ denotes the multiplication in the path algebra $\mathcal{F}$. Multiplication by u_i on the left and on the right endows $\tilde{\mathcal{F}}$ with a structure of $\mathbb{C}^I$–$\mathbb{C}^I$ algebra. This algebra was introduced by Lusztig [Lus00a, 2.1] (see also [Lus98, 2.4] and [Lus00b, 2.1]). It is an associative algebra with unit (given by $\sum_{i \in I} u_i$). Since the Dynkin graph is assumed to be of finite type $\tilde{\mathcal{F}}$ is finitely generated (cf. [Lus00a, Lemma 2.2]).

Let D be a $\mathbb{C}^I$-module, and Z_D be the set of all $\mathbb{C}^I$–$\mathbb{C}^I$ algebra homomorphisms $\pi : \tilde{\mathcal{F}} \to \text{End}_{\mathbb{C}^I}(D)$ (in other words, the set of all representations of $\tilde{\mathcal{F}}$ in D). The set Z_D is naturally an affine variety. Let ϑ' be a regular map

$$\vartheta' : \quad \Lambda_{D,V}^{s,*s} \to Z_D$$

given by

$$\vartheta'((x, p, q)) = \pi \, ,$$

where

$$\pi([h_1 \ldots h_n]) = q_{\text{Out}(h_1)} x_{h_1} \ldots x_{h_n} p_{\text{In}(h_n)}$$

for a path $[h_1 \ldots h_n]$ in $\mathcal{F}$. The map ϑ' being constant on the orbits of G_V, it induces a regular map $\vartheta : \mathfrak{M}_{D,V}^{s,*s} = \Lambda_{D,V}^{s,*s}/G_V \to Z_D$. Let $Z_{D,V} = \vartheta(\mathfrak{M}_{D,V}^{s,*s}) \subset Z_D$.

One has $Z_{D,V} = Z_{D,V'}$ if $\dim V = \dim V'$ and thus the notation $Z_{D,v}$ (where $v = \dim V$) is sometimes used instead of $Z_{D,V}$.

   The following theorem is due to Lusztig.

Theorem.

   A.1.a. [Lus98, Theorem 5.5] [Lus00b, Lemma 4.12c] *The map $\vartheta : \mathfrak{M}^{s,*s}_{D,V} \to Z_{D,V}$ is a homeomorphism both in Zariski and in the smooth topologies.*

   A.1.b. [Lus00b, Lemma 4.12c] *The set $Z_{D,V}$ is locally closed in Z_D.*

   A.1.c. [Lus00b, Lemma 4.12d] $Z_D = \bigsqcup_{v \in \mathbb{Z}_{\geq 0}[I]} Z_{D,v}$.

   The above theorem shows that varieties $\mathfrak{M}^{s,*s}_{D,V}$ for various V can be "glued together" which is crucial for a geometric construction of the tensor product. Nakajima also considered a union of all $\mathfrak{M}^{s,*s}_{D,V}$ for a fixed D (cf. [Nak98], [Nak01a, 2.5]). He calls the resultant variety $\mathfrak{M}_0(\infty, d)$, where $d = \dim D$. It follows from [Lus98, Theorem 5.5] that $\mathfrak{M}_0(\infty, d) = Z_D$.

   Given $\pi \in Z_D$ one can find $v \in \mathbb{Z}_{\geq 0}[I]$ such that $\pi \in Z_{D,v}$ as follows (cf. [Lus98, Section 2]). Let $\dot{D} = \mathcal{F} \otimes_{\mathbb{C}^I} D$. Then $\dot{D}$ is naturally a left $\mathcal{F}$-module. Let $\varpi_\pi \in \mathrm{Hom}_{\mathbb{C}^I}(\dot{D}, D)$ be given by $\varpi_\pi(f \otimes d) = \pi(f)d$, and let $\mathcal{K}_\pi$ be the largest $\mathcal{F}$-submodule of $\dot{D}$ contained in the kernel of ϖ_π. Then $v = \dim(\dot{D}/\mathcal{K}_\pi)$ is finite and $\pi \in Z_{D,v}$.

A.2. **Multiplicity varieties.** Let D, V be $\mathbb{C}^I$-modules, and $\mathbf{D} = (\{0\} = \mathbf{D}^0 \subset \mathbf{D}^1 \subset \ldots \subset \mathbf{D}^n = D)$ be an n-step $\mathbb{C}^I$-filtration of D. Let $\mathbf{v} \in (\mathbb{Z}_{\geq 0}[I])^n$, and $v = \dim V$. Recall (cf. 2.6) that the multiplicity variety $\mathfrak{S}_{D,\mathbf{D},V,\mathbf{v}}$ is a subset of $\mathfrak{M}^{s,*s}_{D,V}$. Consider the restriction of the map $\vartheta : \mathfrak{M}^{s,*s}_{D,V} \to Z_D$ (cf. Appendix A.1) to $\mathfrak{S}_{D,\mathbf{D},V,\mathbf{v}}$. It follows from the definition of the multiplicity variety that this restriction provides a homeomorphism between a multiplicity variety $\mathfrak{S}_{D,\mathbf{D},V,\mathbf{v}}$ and a subvariety $Z_{D,\mathbf{D},v,\mathbf{v}}$ of Z_D consisting of all $\pi \in Z_D$ such that

   - for any $k = 1, \ldots, n$ the $\mathbb{C}^I$-submodule $\mathbf{D}^k$ is an $\tilde{\mathcal{F}}$-submodule of D with respect to the representation $\pi : \tilde{\mathcal{F}} \to \mathrm{End}_{\mathbb{C}^I} D$,
   - $\pi \in Z_{D,v}$,
   - $\pi|_{\mathbf{D}^k/\mathbf{D}^{k-1}} \in Z_{\mathbf{D}^k/\mathbf{D}^{k-1}, \mathbf{v}^k} \subset Z_{\mathbf{D}^k/\mathbf{D}^{k-1}}$ for any $k = 1, \ldots, n$.

This description of the multiplicity varieties is reminiscent of the definition of the Hall-Ringel algebra (cf. [Rin90]) associated to the algebra $\tilde{\mathcal{F}}$. One should be careful however because the set $Z_{D,v} \subset Z_D$ is in general a union of orbits of $\mathrm{Aut}_{\mathbb{C}^I}(D)$, rather than a single orbit. Similar situation occurs in the geometric (quiver) construction of the positive part of a (quantum) universal enveloping algebra when the underlying quiver is not of finite type (cf. [Lus91, Sch91]).

A.3. **The tensor product diagram.** Let D, D^1, $\ldots$, D^n be $\mathbb{C}^I$-modules. By analogy with Lusztig's construction of the canonical basis one can consider the following diagram (cf. [Lus90, 6.1a]).

(A.3.a)

$$
\begin{array}{ccc}
 & Z' \xrightarrow{p_2} Z'' & \\
 {\scriptstyle p_1}\swarrow & & \searrow {\scriptstyle p_3} \\
Z_{D^1} \times \ldots \times Z_{D^n} & & Z_D
\end{array}
$$

Here the notation is as follows:

Z'' is the variety of all pairs $(\pi, \mathbf{W})$ consisting of $\pi \in Z_D$ and a $\mathbb{C}^I$-filtration $\mathbf{W} = (0 = \mathbf{W}^0 \subset \mathbf{W}^1 \subset \ldots \subset \mathbf{W}^n = D)$ such that $\pi(f)\mathbf{W}^k \subset \mathbf{W}^k$ for any $f \in \mathcal{F}$, $k = 1, \ldots, n$, and $\dim(\mathbf{W}^k/\mathbf{W}^{k-1}) = \dim D^k$,

Z' is the variety of all triples $(\pi, \mathbf{W}, \mathbf{R})$ where $(\pi, \mathbf{W}) \in Z''$, and $\mathbf{R}$ is an n-tuple $(\mathbf{R}^1, \ldots, \mathbf{R}^n)$ consisting of $\mathbb{C}^I$-isomorphisms $\mathbf{R}^k : D^k \xrightarrow{\sim} \mathbf{W}^k/\mathbf{W}^{k-1}$ for $k = 1, \ldots, n$,

$p_2((\pi, \mathbf{W}, \mathbf{R})) = (\pi, \mathbf{W})$,

$p_3((\pi, \mathbf{W})) = \pi$,

$p_1((\pi, \mathbf{W}, \mathbf{R})) = (\pi^1, \ldots, \pi^n)$, where $\pi^k = (\mathbf{R}^k)^{-1}(\pi|_{\mathbf{W}^k/\mathbf{W}^{k-1}})\mathbf{R}^k$ for any $k = 1, \ldots, n$.

The diagram (A.3.a) is closely related to tensor product and multiplicity varieties. In particular a subset

$$\text{(A.3.b)} \qquad p_2 \circ p_1^{-1}(Z_{D^1, \mathbf{v}^1} \times \ldots \times Z_{D^n, \mathbf{v}^n}) \bigcap p_3^{-1}(Z_{D,V}) \subset Z''$$

is the total space of a fibration over the space of $\mathbb{C}^I$-filtrations of D with dimensions of the subfactors given by $\dim D^1, \ldots, \dim D^n$, and the fiber of this fibration over a point $\mathbf{D}$ is isomorphic to the multiplicity variety ${}^n\mathfrak{S}_{D, \mathbf{D}, V, \mathbf{v}}$. It follows that the subset (A.3.b) has pure dimension and the number of its irreducible components is equal to

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathfrak{g}'}\left(L(d, v), L(\mathbf{d}^1, \mathbf{v}^1) \otimes \ldots \otimes L(\mathbf{d}^n, \mathbf{v}^n)\right),$$

where $d = \dim D$, $\mathbf{d}^k = \dim D^k$, $v = \dim V$, and $L(d, v)$ denotes the highest weight irreducible representation of $\mathfrak{g}'$ with the highest weight $(d - v + Xv, v)$ (cf. 3.11). Tensor product varieties also can be described in the context of the diagram (A.3.a). They are related to a resolution of singularities of Z''.

Let $Z = \bigsqcup Z_D$, where D ranges over (representatives of the) isomorphism classes of $\mathbb{C}^I$-modules. The diagram (A.3.a) can be (conjecturally) used to equip the category of sheaves on Z perverse with respect to the stratification $Z_D = \bigsqcup Z_{D,V}$ with a structure of a Tannakian category. The results of this paper concerning crystal tensor product provide a step toward finding a relation between this category and the category of integrable finite dimensional representations of the Lie algebra $\mathfrak{g}'$.

References

[AHDM78] M. Atiyah, N. Hitchin, V. Drinfeld, and Yu. Manin, *Construction of instantons*, Phys. Lett. A **65** (1978), no. 3, 185–187.

[BG99] A. Braverman and D. Gaitsgory, *Crystals via the affine grassmannian*, 1999, arXiv:math.AG/9909077.

[BM83] W. Borho and R. MacPherson, *Partial resolutions of nilpotent varieties*, Analysis and topology on singular spaces, II, III (Luminy, 1981), Soc. Math. France, Paris, 1983, pp. 23–74.

[CB00] W. Crawley-Boevey, *Geometry of the moment map for representations of quivers*, 2000, available at *http://www.amsta.leeds.ac.uk/~pmtwc/*, to appear in Compositio Math.

[Gin91] V. Ginzburg, *Lagrangian construction of the enveloping algebra $U(\mathrm{sl}_n)$*, C. R. Acad. Sci. Paris Sér. I Math. **312** (1991), no. 12, 907–912.

[GL92] I. Grojnowski and G. Lusztig, *On bases of irreducible representations of quantum gl_n*, Kazhdan-Lusztig theory and related topics (Chicago, IL, 1989), AMS, Providence, RI, 1992, pp. 167–174.

[Hal59] P. Hall, *The algebra of partitions*, Proc. 4th Canadian Math. Congress (Banff), University of Toronto Press, 1959, pp. 147–159.

[Jos95]   A. Joseph, *Quantum groups and their primitive ideals*, Springer-Verlag, Berlin, 1995.

[Kas90]   M. Kashiwara, *Crystalizing the q-analogue of universal enveloping algebras*, Comm. Math. Phys. **133** (1990), no. 2, 249–260.

[Kas91]   M. Kashiwara, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), no. 2, 465–516.

[Kas94]   M. Kashiwara, *Crystal bases of modified quantized enveloping algebra*, Duke Math. J. **73** (1994), no. 2, 383–413.

[KL87]   D. Kazhdan and G. Lusztig, *Proof of the Deligne-Langlands conjecture for Hecke algebras*, Invent. Math. **87** (1987), no. 1, 153–215.

[KN90]   P. Kronheimer and H. Nakajima, *Yang-Mills instantons on ALE gravitational instantons*, Math. Ann. **288** (1990), no. 2, 263–307.

[KS97]   M. Kashiwara and Y. Saito, *Geometric construction of crystal bases*, Duke Math. J. **89** (1997), no. 1, 9–36.

[Lus90]   G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), no. 2, 447–498.

[Lus91]   G. Lusztig, *Quivers, perverse sheaves, and quantized enveloping algebras*, J. Amer. Math. Soc. **4** (1991), no. 2, 365–421.

[Lus98]   G. Lusztig, *On quiver varieties*, Adv. Math. **136** (1998), no. 1, 141–182.

[Lus00a]   G. Lusztig, *Constructible functions on varieties attached to quivers*, 2000, available at *http://www-math.mit.edu/~gyuri/pub.html*.

[Lus00b]   G. Lusztig, *Quiver varieties and Weyl group actions*, Ann. Inst. Fourier (Grenoble) **50** (2000), no. 2, 461–489.

[Mac95]   I. Macdonald, *Symmetric functions and Hall polynomials*, second ed., The Clarendon Press Oxford University Press, New York, 1995.

[Mal00]   A. Malkin, *Tensor product varieties and crystals. GL(N) case.*, arXiv:math.AG/0103026.

[Mal01a]   A. Malkin, *Another generalization of Hall polynomials to ADE case*, in preparation.

[Nak94]   H. Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. **76** (1994), no. 2, 365–416.

[Nak98]   H. Nakajima, *Quiver varieties and Kac-Moody algebras*, Duke Math. J. **91** (1998), no. 3, 515–560.

[Nak01a]   H. Nakajima, *Quiver varieties and finite dimensional representations of quantum affine algebras*, J. Amer. Math. Soc. **14** (2001), no. 1, 145–238.

[Nak01b]   H. Nakajima, *Quiver varieties and tensor products*, arXiv:math.QA/0103008.

[Rin90]   C. M. Ringel, *Hall algebras*, Topics in algebra, Part 1 (Warsaw, 1988), PWN, Warsaw, 1990, pp. 433–447.

[Sch91]   A. Schofield, *Notes on constructing Lie algebras from finite-dimensional algebras*, 1991, preprint.

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